New Trends In High Energy Physics

Symplectic Field Theory of the Galilean Covariant Scalar and Spinor Representations

Gustavo Xavier Antunes Petronilo, A.E. Santana, S. Ulhoa

International Center of Physics
Institute of Physics - University of Brasília, DF, Brazil

May 16 2019
New Trends in High Energy Physics

Sumário

1. Motivation
2. Galilean Covariance
3. Hilbert Space and Symplectic Structure
4. Symplectic Quantum Mechanics and the Galilean Covariance
5. Spin 1/2 Symplectic Representation
   Electromagnetic Interactions
6. Conclusion
New Trends in High Energy Physics

Motivation

- Wigner Function $\rightarrow$ Landau Problem
  - Galilean Covariance
  - Star Product
  - Symplectic Representations

---

Galilei Group

Galilei Transformations

\[
\begin{align*}
    x' &= Rx + vt + a \\
    t' &= t + b
\end{align*}
\]

- \( R \rightarrow \) rotations (3 parameters)
- \( v \rightarrow \) Galilei’s boosts (3 parameters)
- \( a \rightarrow \) translations (3 parameters)
- \( b \rightarrow \) clock synchronization (1 parameter)
New Trends in High Energy Physics

Galilei Group

We denote the transformations of Galilei by

\[ G(x, t) = (x', t'), \]

where \( G \) is given by

\[ G = (b, a, v, R), \]
New Trends in High Energy Physics

Galilei Group

\[ G_1 = (b_1, a_1, v_1, R_1) \]
\[ G_2 = (b_2, a_2, v_2, R_2) \]

\( G_2 G_1 \), results in a composition law given by

\[ G_1 G_2 = (b_1 + b_2, a_2 + R_2 a_1 + b_1 v_2, v_2, R_2 v_1, R_2 R_1). \]
\[ G_2 G_1 = G \) (is also a Galilei’s transformation).

Identity transformation

\[ E = (0, 0, 0, 1) \]

Inverse

\[ G^{-1} = (-b, -R^{-1}(a - bv), -R^{-1}v, R^{-1}) \]
Consider a particle free of mass $m$, the ratio of dispersion is given by

$$\mathbf{p}^2 - 2mE = 0$$

We can then define a 5-vector, $p^\mu = (p_x, p_y, p_z, m, E) = (p^i, m, E)$, with $i = 1, 2, 3$.

Thus, to define a scalar product of the type $p\mu p_\nu g^{\mu\nu} = \mathbf{p}^2 - 2mE = k^2$

$$p_\mu p_\nu g^{\mu\nu} = p_ip_i - p_4p_5 - p_5p_4 = \mathbf{p}^2 - 2mE = k^2,$$

where $g^{\mu\nu}$ is the metric of Space to be constructed, and $p_\nu g^{\mu\nu} = p_\mu$. 
Let $q^\mu$ the set of canonical coordinates associated with $p^\mu$, we write

$q^\mu = (q, q^4, q^5)$.

- $q$ is the canonical coordinate associated with $p$;
- $q_4$ is the canonical coordinate associated with $E$, and thus can be considered as the time coordinate;
- $q_5$ is the canonical coordinate associated with $m$ explicitly given in terms of $q$ e $q^4$

$q^\mu q_\mu = q_\mu q_\nu g^{\mu\nu} = q - 2q_4q_5 = s^2$. Since $p^\mu p_\mu = 0$, we have to take $s = 0$

$q^5 = \frac{q^2}{2t}$; or infinitesimally, we obtain $\delta q^5 = \nu \cdot \delta q_2 $.

Therefore, the fifth component is defined by velocity.
New Trends in High Energy Physics

Galilean Covariance

That can be seen as a special case of a vector product in $G$ denoted as

$$(x|y) = g^\mu_\nu x_\mu y_\nu = \sum_{i=1}^{3} x_i y_i - x_4 y_5 - x_5 y_4,$$

where $x^4 = y^4 = t$, $x^5 = \frac{x^2}{2t}$ e $y^5 = \frac{y^2}{2t}$

we can introduce the metric

$$ (g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}.$$
Consider $\mathcal{G}$ an analytical manifold where each point is specified by coordinates $q_\mu$, with $\mu = 1, 2, 3, 4, 5$ and metric specified by (9). The coordinates of each point in the cotangent-bundle $T^*\mathcal{G}$ will be denoted by $(q_\mu, p_\mu)$. The Space $T^*\mathcal{G}$ is equipped with a symplectic structure via a 2-form.

$$\omega = dq^\mu \wedge dp_\mu$$

called the symplectic form (sum over repeated indices is assumed). We consider the following bidifferential operator on $C^\infty(T^*\mathcal{G})$ functions,

$$\Lambda = \frac{\langle \partial_\mu \partial \rangle}{\partial q_\mu \partial p_\mu} - \frac{\langle \partial_\nu \partial \rangle}{\partial p_\nu \partial q_\mu}$$
such that for $C^\infty$ functions, $f(q, p)$ and $g(q, p)$, we have

$$\omega(f\Lambda, g\Lambda) = f\Lambda g = \{f, g\}$$

where

$$\{f, g\} = \frac{\partial f}{\partial q^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p^\mu} \frac{\partial g}{\partial q_\mu}$$

is the poison bracket and $f\Lambda$ and $g\Lambda$ are two vector fields given by $h\Lambda = X_h = -\{h, \}$. The Space $T^*G$ endowed with this symplectic structure is called the phase-space, and will be denoted by $\Gamma$. 
To associate the Hilbert Space with the phase-space $\Gamma$, we will consider the set of complex functions of integrable square, $\phi(q, p)$ in $\Gamma$, such that

$$\int dp dq \phi^\dagger(q, p) \phi(q, p) < \infty$$

is called the real bilinear form. In this case $\phi(q, p) = \langle q, p|\phi \rangle$ is written with the aid of

$$\int dp dq |q, p\rangle\langle q, p| = 1$$

where $\langle \phi| \text{ is the dual vector of } |\phi\rangle$. This symplectic Hilbert Space is denoted by $H(\Gamma)$. 
Consider the unit transformations $U: \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ such that $\langle \psi_1 | \psi_2 \rangle$ is invariant. Using the $\Lambda$ operator, we define a mapping $e^{i \frac{\Lambda}{2}} = \star: \Gamma \times \Gamma \to \Gamma$ called as Moyal (or star) product, defined by.

\[
f \star g = f(q, p) \exp \left[ \frac{i}{2} \left( \frac{\overleftarrow{\partial}}{\partial q^\mu} \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu} \frac{\overrightarrow{\partial}}{\partial q^\mu} \right) \right] g(q, p)
\]

where $\hbar = 1$. 

The generators of $U$ can be introduced by the following (Moyal-Weyl) star-operators:

$$\hat{F} = f(q, p)\star = f \left( q^\mu + \frac{i}{2} \frac{\partial}{\partial p_\mu}, p^\mu - \frac{i}{2} \frac{\partial}{\partial q_\mu} \right).$$
To construct a representation of Galilei algebra in $\mathcal{H}$, we define the following operators,

$$
\hat{P}^\mu = p^\mu \ast = p^\mu - \frac{i}{2} \frac{\partial}{\partial q_\mu},
$$

$$
\hat{Q}^\mu = q \ast = q^\mu + \frac{i}{2} \frac{\partial}{\partial p_\mu}.
$$

and

$$
\hat{M}^\mu_{\nu\sigma} = M^\mu_{\nu\sigma} \ast = \hat{Q}_\nu \hat{P}_\sigma - \hat{Q}_\sigma \hat{P}_\nu.
$$
From this set of unitary operators we obtain, after some simple calculations, the following set of commutations relations,

\[
\begin{align*}
\left[ \hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma} \right] &= -i\left( g_{\nu\rho} \hat{M}_{\mu\sigma} - g_{\mu\rho} \hat{M}_{\nu\sigma} + g_{\mu\sigma} \hat{M}_{\nu\rho} - g_{\mu\sigma} \hat{M}_{\nu\rho} \right), \\
\left[ \hat{P}_\mu, \hat{M}_{\rho\sigma} \right] &= -i\left( g_{\mu\rho} \hat{P}_\sigma - g_{\mu\sigma} \hat{P}_\rho \right), \\
\left[ \hat{P}_\mu, \hat{P}_\sigma \right] &= 0.
\end{align*}
\]

A Casimir invariant of this algebra is \( \hat{P}_\mu \hat{P}_\mu \).
Consider a vector \( q^\mu \in G \) that obeys the set of linear transformations of the type

\[
\bar{q}^\mu = G^\mu_\nu q^\nu + a^\mu .
\]

A particular case of interest of these transformation, given by

\[
\begin{align*}
\bar{q}^i &= R^i_j q^j + v^i q^4 + a^i \\
\bar{q}^4 &= q^4 + a^4 \\
\bar{q}^5 &= q^5 - (R^i_j q^j) v_i + \frac{1}{2} v^2 q^4.
\end{align*}
\]
In the matricial form, the homogeneous transformations are written as

\[ G^\mu_{\nu} = \begin{pmatrix} R_1^1 & R_2^1 & R_3^1 & v^i & 0 \\ R_1^2 & R_2^2 & R_3^2 & v^2 & 0 \\ R_1^3 & R_2^3 & R_3^3 & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v_i R_j^i & v_i R_2^i & v_i R_3^i & \frac{v^2}{2} & 1 \end{pmatrix} \]
We can write down the generators as

\[
\hat{J}_i = \frac{1}{2} \epsilon_{ijk} \hat{M}_{jk}
\]

\[
\hat{K}_i = \hat{M}_{5i}
\]

\[
\hat{C}_i = \hat{M}_{4i}
\]

\[
\hat{D}_i = \hat{M}_{5i}
\]
With the commutation relations, considering non-vanish ones, they are rewritten as

\[
\begin{align*}
\left[ \hat{J}_i, \hat{J}_j \right] &= i\epsilon_{ijk} \hat{J}_k \\
\left[ \hat{J}_i, \hat{C}_j \right] &= i\epsilon_{ijk} \hat{C}_k \\
\left[ \hat{D}, \hat{K}_i \right] &= i\hat{K}_i \\
\left[ \hat{P}_4, \hat{D} \right] &= i\hat{P}_4 \\
\left[ \hat{P}_i, \hat{K}_j \right] &= i\delta_{ij} \hat{P}_5 \\
\left[ \hat{P}_4, \hat{K}_i \right] &= i\hat{P}_i \\
\left[ \hat{D}, \hat{P}_5 \right] &= i\hat{P}_5
\end{align*}
\]

\[
\begin{align*}
\left[ \hat{J}_i, \hat{K}_j \right] &= i\epsilon_{ijk} \hat{K}_k \\
\left[ \hat{K}_i, \hat{C}_j \right] &= i\delta_{ij} \hat{D} + i\epsilon_{ijk} \hat{J}_k \\
\left[ \hat{C}_i, \hat{D} \right] &= i\hat{C}_i \\
\left[ \hat{J}_i, \hat{P}_j \right] &= i\epsilon_{ijk} \hat{P}_k \\
\left[ \hat{P}_i, \hat{C}_j \right] &= i\delta_{ij} \hat{P}_4 \\
\left[ \hat{P}_5, \hat{C}_i \right] &= i\hat{P}_i
\end{align*}
\]
This relations form an algebra which has as subalgebra the Lie algebra of Galilei group in the case of $\mathbb{R}^3 \times t$, considering $J_i$ the generators of rotations and $C_i$ of the pure Galilei transformations, $P_\mu$ the spacial and temporal translations and $D$ of the kind temporal dilation (which we will not discuss here). The commutation of $K_i$ and $P_i$ is naturally non-zero in this context, being $P_5$ will be related with mass. The invariants of this algebra are

\[
I_1 = \hat{P}_\mu \hat{P}^\mu
\]

\[
I_2 = \hat{P}_5
\]
Using the Casimir invariants $I_1$ and $I_2$ and applying in $\Psi$,

\[
\hat{P}_\mu \hat{P}_\mu \Psi = k^2 \Psi \\
\hat{P}_5 \Psi = -m \Psi
\]

From this we obtain

\[
\left(p^2 - ip \cdot \nabla - \frac{1}{4} \nabla^2 - k^2\right) \Psi = 2 \left(p_4 - \frac{i}{2} \partial_t\right) \left(p_5 - \frac{i}{2} \partial_5\right) \Psi,
\]

with $\hat{P}^\mu = p^\mu \star = p^\mu - \frac{i}{2} \frac{\partial}{\partial q_\mu}$. A solution for this equation is

\[
\Psi = e^{-2i[(p_5+m)q_5+(p_4+E)t]} \Phi(q, p).
\]

Thus,

\[
\frac{1}{2m} \left(p^2 - ip \cdot \nabla - \frac{1}{4} \nabla^2\right) \Phi = \left(E + \frac{k^2}{2m}\right) \Phi,
\]
This equation, and its complex conjugate, can also be obtained by the Lagrangian density in phase-Space (we use $d^\mu = d/dq_\mu$)

$$
\mathcal{L}_0 = \partial^\mu \psi(q, p) \partial \psi^*(q, p) + \frac{i}{2} p^\mu [\psi(q, p) \partial^\mu \psi^*(q, p) \\
- \psi^*(q, p) \partial^\mu \psi(q, p)] + \left[ \frac{p^\mu p_\mu}{4} - k^2 \right] \psi = 0.
$$

The association of this representation with the Wigner formalism is given by

$$
f_w(q, p) = \Psi(q, p) \star \Psi^\dagger(q, p)
$$

where $f_w(q, p)$ is the Wigner function.
In order to study the representations of spin particles 1/2, we will introduce the \( \gamma^\mu \hat{P}_\mu \), where \( \hat{P}_\mu = p_\mu - \frac{i}{2} \partial_\mu \) in such a way that acting on the 5-spinor in the phase-space \( \Psi(q, p) \), we have

\[
\left( \gamma^\mu \hat{P}_\mu - k \right) \Psi(p, q) = 0
\]

or

\[
\gamma^\mu \left( p_\mu - \frac{i}{2} \partial_\mu \right) \Psi(p, q) = k\Psi(p, q)
\]

Which is the galilean covariant Pauli-Schrödinger equation in phase-space.
Consequently the mass layer condition is obtained by following the usual steps.

\[(\gamma^\mu \hat{P}_\mu)(\gamma^\nu \hat{P}_\nu)\psi(q, p) = k^2 \psi(q, p),\]

therefore

\[\gamma^\mu \gamma^\nu (\hat{P}_\mu \hat{P}_\nu) = k^2 = \hat{P}_\mu \hat{P}_\nu,\]
since \( \hat{P}_\mu \hat{P}_\nu = \hat{P}_\nu \hat{P}_\mu \), we have

\[
\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \hat{P}_\nu \hat{P}_\mu = \hat{P}^\mu \hat{P}^\nu,
\]

so

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu \nu}
\]
This equation can be derive from the Lagrangian density for spin 1/2 particles in phase-Space, which is given by

\[ \mathcal{L} = -\frac{i}{4} \left( (\partial_\mu \bar{\Psi}) \gamma^\mu \Psi - \bar{\Psi} (\gamma^\mu \partial_\mu \Psi) \right) - (k - \gamma^\mu p_\mu) \Psi \bar{\Psi}. \]

where \( \bar{\Psi} = \Psi^\dagger \zeta \), with

\[ \zeta = -\frac{i}{\sqrt{2}} \{ \gamma^4 + \gamma^5 \} \]

In the case of Pauli-Schrödinger in phase-space equation the association with the Wigner function is given by

\[ f_w = \Psi \star \bar{\Psi}, \]
Let us examine the gauge symmetries in phase-space demanding
the invariance of the Lagrangian by a local gauge transformation
given by $e^{\Lambda(q,p)}\Psi$. This leads to the minimal coupling,

$$\hat{P}_\mu \Psi \to \left(\hat{P}_\mu - e\hat{A}_\mu\right) \Psi = \left(p_\mu - \frac{i}{2} \partial_\mu - e\hat{A}_\mu\right) \Psi,$$
This describes an electron in an external field, with the Pauli-Schrödinger equation in phase-space given by

\[
\gamma^\mu \left( p_\mu - \frac{i}{2} \partial_\mu - e\hat{A}_\mu \right) - k \right] \Psi = 0.
\]

In order to illustrate such result, lets consider a electron in a external field given by \( \hat{A}_\mu(\hat{A}, \hat{A}_4, \hat{A}_5) \), with \( \hat{A}_4 = -\phi \) and \( \hat{A}_5 = 0 \).
Considering the following representation of $\gamma^{\mu}$ matrices

\[
\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}.
\]

where $\sigma^i$ are the Pauli matrices and $\sqrt{2}$ is the identity matrix $2 \times 2$ multiplied by $\sqrt{2}$. We can rewrite the object $\Psi$, as

\[
\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix},
\]

where $\varphi$ and $\chi$ are 2-spinors dependent on $x^\mu; \mu = 1, \ldots, 5$. 
Thus, in the representation where $k = 0$,

\[
\sigma \cdot \left( p - \frac{i}{2} \partial q - e \hat{A} \right) \varphi - \sqrt{2} \left( p_5 - \frac{i}{2} \partial 5 \right) \chi = 0,
\]

\[
\sqrt{2} \left( p_4 - \frac{i}{2} \partial_t - e \phi \right) \varphi - \sigma \cdot \left( p - \frac{i}{2} \partial q - e \hat{A} \right) \chi = 0.
\]
Solving the coupled equations we get an equation for $\varphi$ and $\chi$, given by

$$
\left[ \sigma \cdot \left( p - \frac{i}{2} \partial_q - e \hat{A} \right) \right]^2 \varphi = 2 \left( p_4 - \frac{i}{2} \partial_t - e \phi \right) \left( p_5 - \frac{i}{2} \partial_5 \right) \varphi
$$

$$
\left[ \sigma \cdot \left( p - \frac{i}{2} \partial_q - e \hat{A} \right) \right]^2 \chi = 2 \left( p_4 - \frac{i}{2} \partial_t - e \phi \right) \left( p_5 - \frac{i}{2} \partial_5 \right) \chi
$$

with

$$
f_w = \psi \ast \bar{\psi},
$$

$$
= i \varphi \ast \chi^\dagger - i \chi \ast \varphi^\dagger
$$
Replacing the eigenvalues of \( \hat{P}_4 \) and \( \hat{P}_5 \), we have

\[
\left[ \frac{1}{2m} \left( \mathbf{\sigma} \cdot \left( \mathbf{p} - \frac{i}{2} \partial_q - e \hat{\mathbf{A}} \right) \right)^2 + e\phi \right] \varphi = E\varphi
\]

\[
\left[ \frac{1}{2m} \left( \mathbf{\sigma} \cdot \left( \mathbf{p} - \frac{i}{2} \partial_q - e \hat{\mathbf{A}} \right) \right)^2 + e\phi \right] \chi = E\chi
\]

Which is the non-covariant form of the time independent Pauli-Schrödinger equation in phase-space.
Laundau levels

\[ E = \frac{eB}{m} \left( n + \frac{1}{2} - \frac{s}{2} \right) - \frac{k^2}{2m} \]

where \( s = \pm 1 \). and, again

\[ f_w = \Psi \star \bar{\Psi}, \]
\[ = i\varphi \star \chi^\dagger - i\chi \star \varphi^\dagger \]
New Trends in High Energy Physics
Electromagnetic Interactions

Quasi-amplitudes of probability
New Trends in High Energy Physics

Electromagnetic Interactions

Wigner Functions
We study the spin 1/2 particle equation, the Pauli-Schrödinger equation, in the context of Galilean covariance and then construct a phase-space formalism using such covariance. We construct the formalism of the quantum mechanics of the Galilean covariant phase-space and we arrive at the representations of the spin 0 and spin 1/2 equations, where for the spin equation 1/2, the Dirac-like equation, we study the electron in an external field and with the supposed solution we were able to recover the Pauli-Schrödinger equation (written its non-covariant form) in phase-space.
This work was supported by CAPES. 

The Brazilian universities and research centers are under attack, with cutting of budget, by the present days Brazilian Federal government. Due to this situation works like this may be impossible.
References

Inönü and Wigner, IL NUOVO CIMENTO 9, (1952) 706.


References


de Melo, GR; de Montigny, M; Pompeia, PJ; Santos, Esdras S

Amorim, RGG; Khanna, FC; Malbouisson, APC; Malbouisson, JMC; Santana, AE
Thank you!