Effective Hadronic Supersymmetry based on Octonions

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Symmetry is a wide-reaching concept that has been used in variety ways in physics. Originally it was used mainly to describe the arrangement of atoms in molecules and crystals (geometric symmetries.)

In the course of 20th century and beyond, it is considerably extended and covers some of the most fundamental ideas in physics.

1. Kinematic (space-time) symmetries.Examples are rotational invariance in non-relativistic quantum mechanics

$$H = \frac{p^2}{2m} + V(r)$$

$$H\psi(\vec{r}) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r})$$

leading to SO(3) symmetry, and Lorentz invariance in relativistic quantum mechanics

$$[\gamma_\mu(i\partial_\mu - eA_\mu) + m]\psi(\vec{r},t) = 0$$

which leads to SO(3,1) symmetry.

2. Dynamic (internal) supersymmetries.

Here we see development of two ideas:

a. there may exist in nature other symmetries in addition to space-time.

b. There may be symmetries of dynamical origin, related to special properties of the Hamiltonian (or Lagrangean) operator, rather than its space-time behavior.

Supersymmetry:

In normal symmetry, symmetry operations transform separately fermions into fermions, bosons into bosons.

In supersymmetry, some of the symmetry operations transform bosons into fermions and vice versa. Introduction of SUSY led to other major developments in physics. SUSY is used in variety of ways. Particularly important are:

1. Kinematic (space-time) supersymmetries:
For example Wess-Zumino invariance. No experimental evidence for it yet.

2 Dynamic (internal) supersymmetries.

Atomic Physics

Pauli (1926) $SO(4)$

Heisenberg (1932) $SU_T(2)$

Wigner (1937) $SU_{TS}(4) \supset SU_T(2) \otimes SU_S(2)$

Particle Physics

Nuclear Physics

Elliot (1958) $SU(3)$

Gell-Mann and Ne'eman (1962) $SU(3)$

Gürsey and Radicati (1964) $SU_{sf}(6) \supset SU_f(3) \otimes SU2)$

Arima and Iachello (1974) SU(6)

A Brief History of Dynamic SUSY in Physics

Miyazawa (1966) $SU(6/21)$

Particle Physics

Nuclear Physics

Balantekin, Bars & Iachello (1978) $SU(6/\Omega)$

Catto & Gürsey (1984) $SU(6/21)$

Is color related to octonions?

Is the quark structure a consequence of octonionic quantum mechanics?

Some consequences:

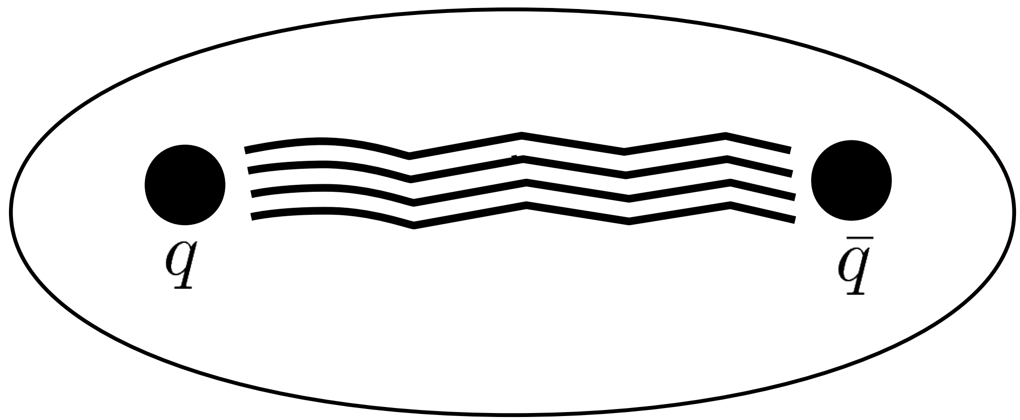
Since G_2 is the automorphism group of octonions

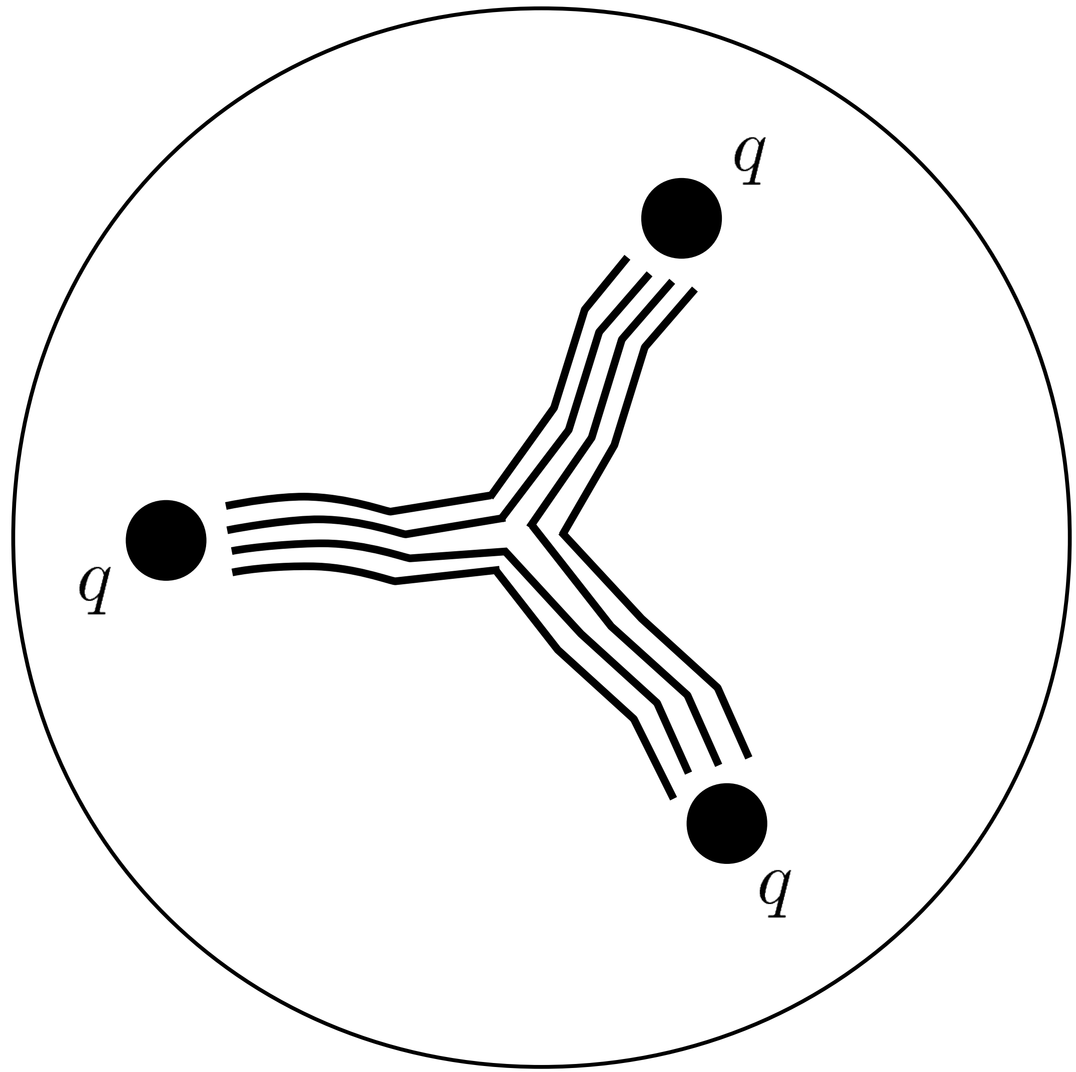
$G_2 = Aut(\Omega)$

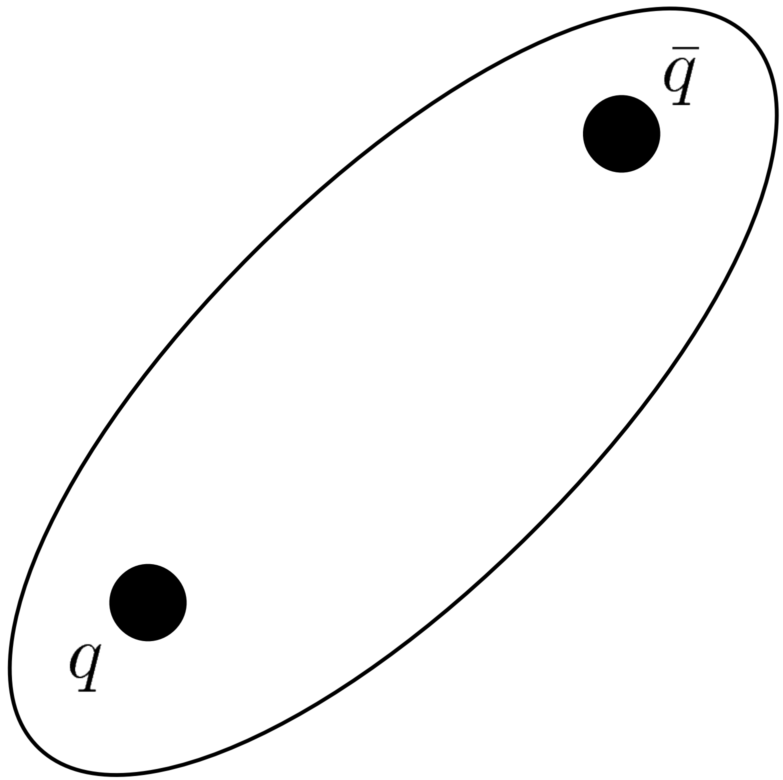
and it can be imbedded into $SO(7)$

$SO(7) \supset G_2 \supset SU(3)$

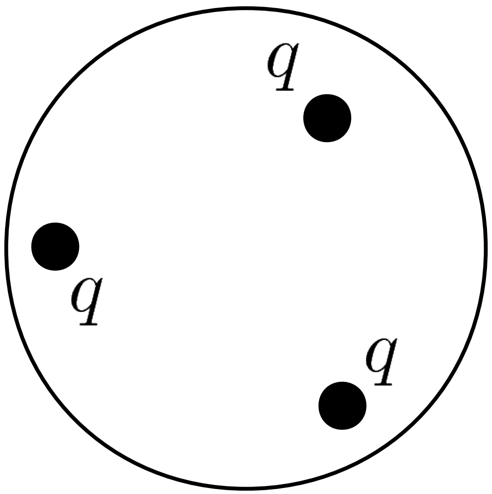
is $SO(7)$ a higher symmetry of strong interactions?

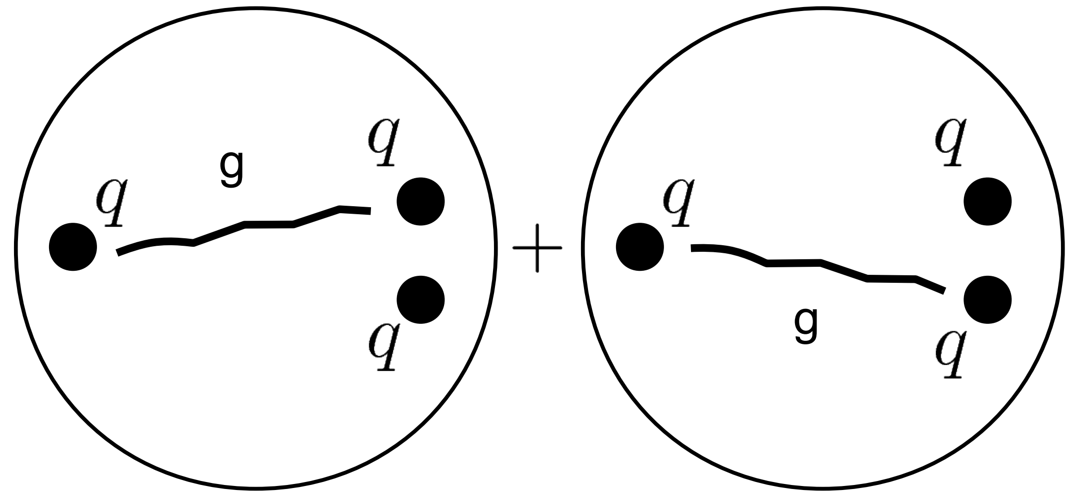

q
$\bar{q}$

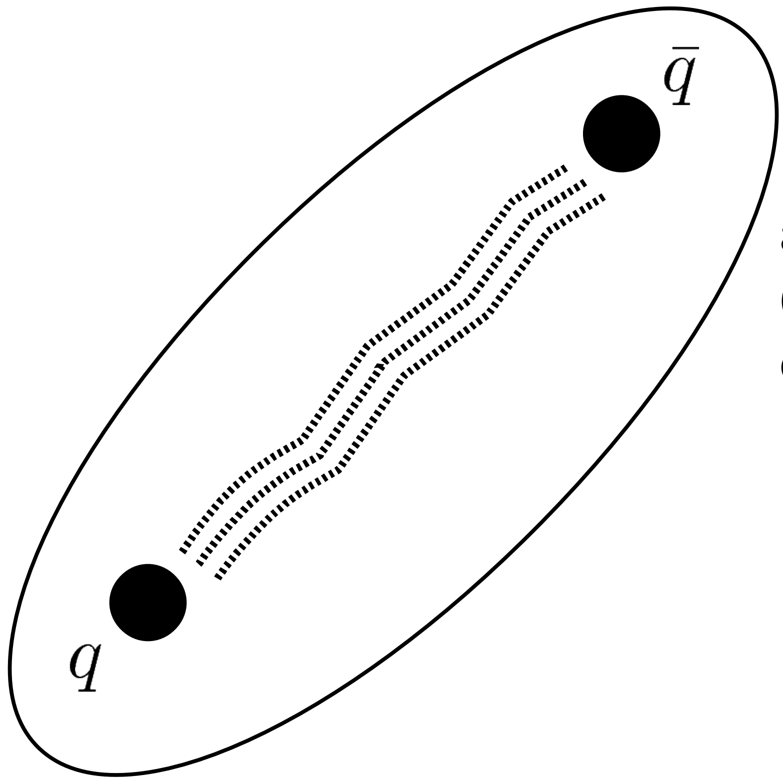

q
q
q



$\bar{q}$
q

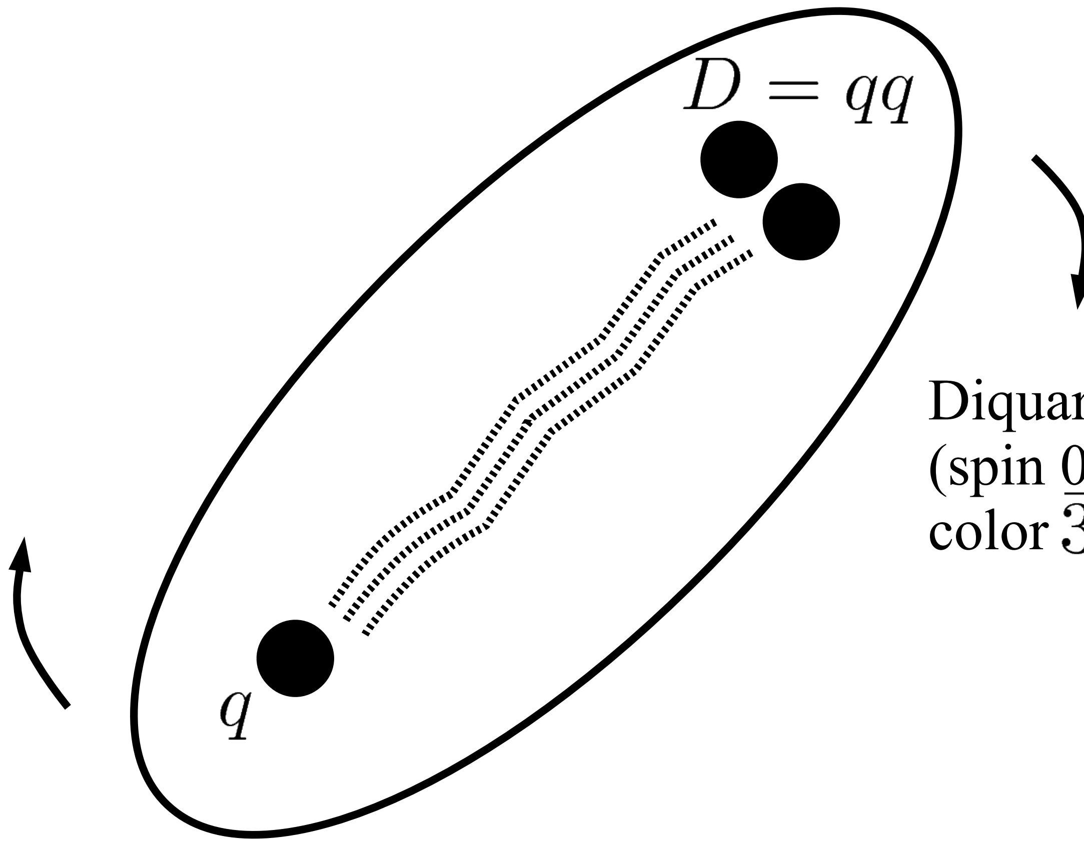

q
q
q


q
g
q
q
+
q
q
q
g



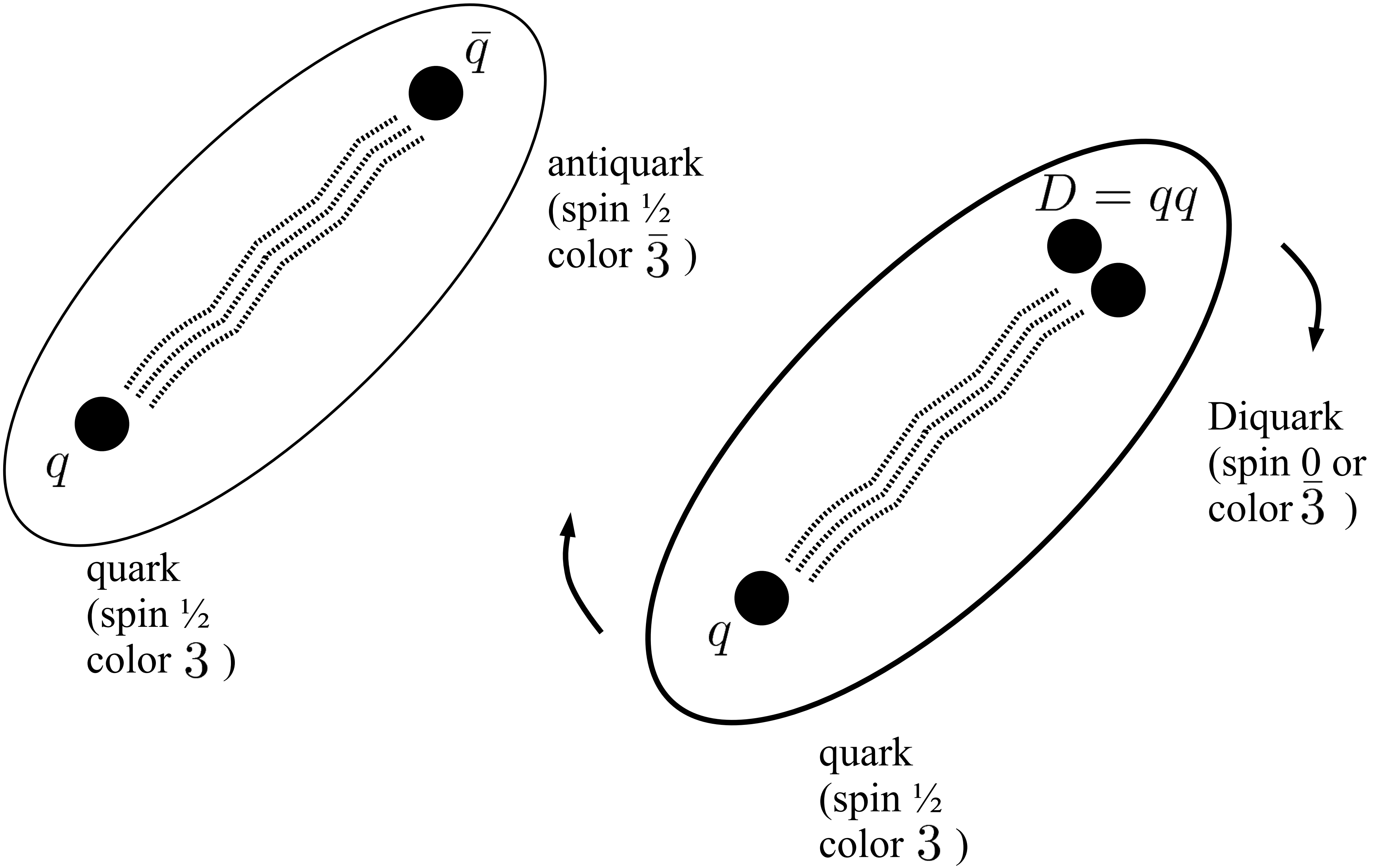
antiquark
(spin ½
color $\bar{3}$)

quark
(spin ½
color 3)



Diquark
(spin 0 or 1
color $\bar{3}$)

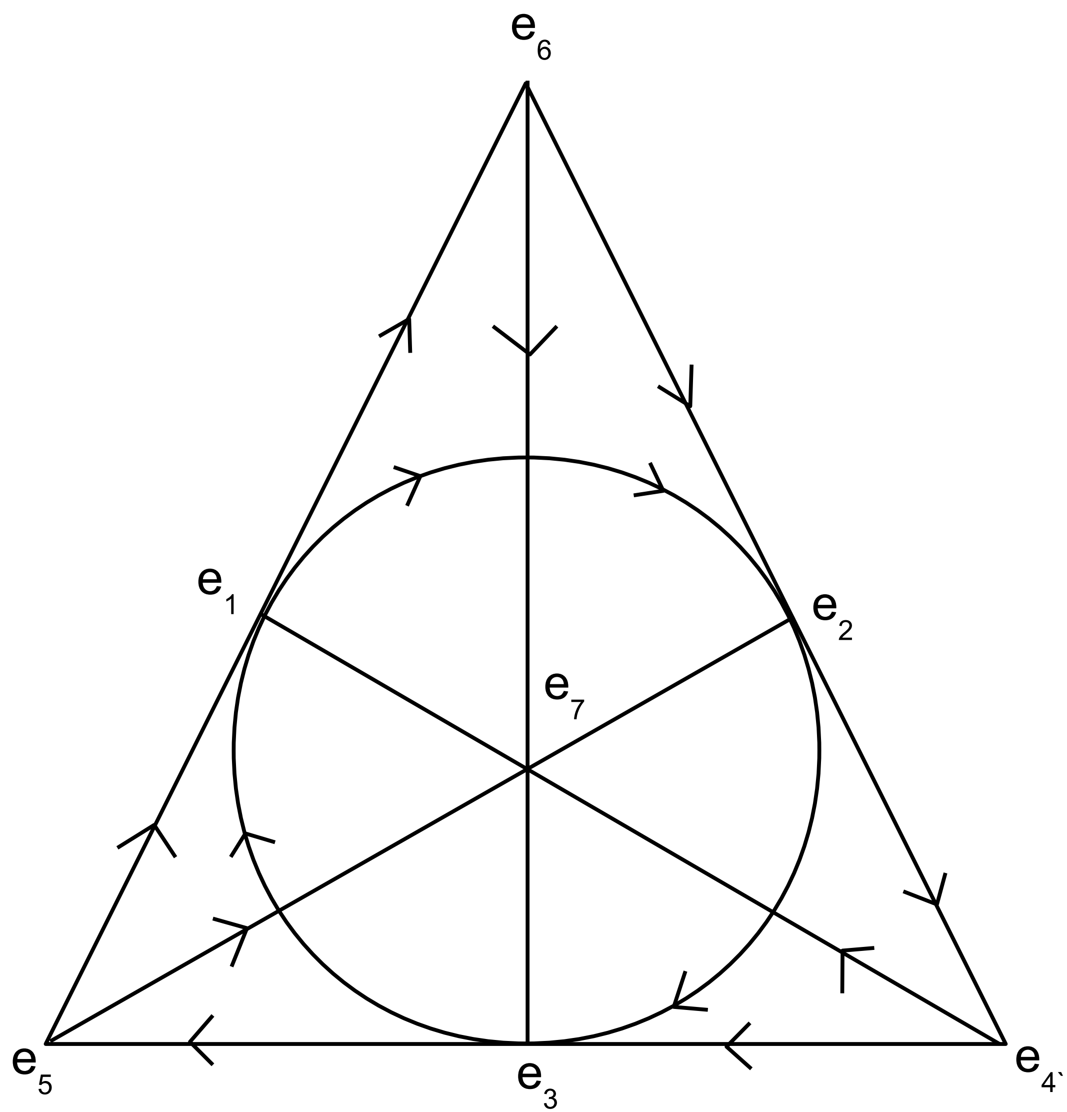
quark
(spin ½
color 3)



$\bar{q}$
antiquark
(spin ½
color $\bar{3}$)
q
quark
(spin ½
color 3)
$D = qq$
Diquark
(spin 0 or 1
color $\bar{3}$)
q
quark
(spin ½
color 3)

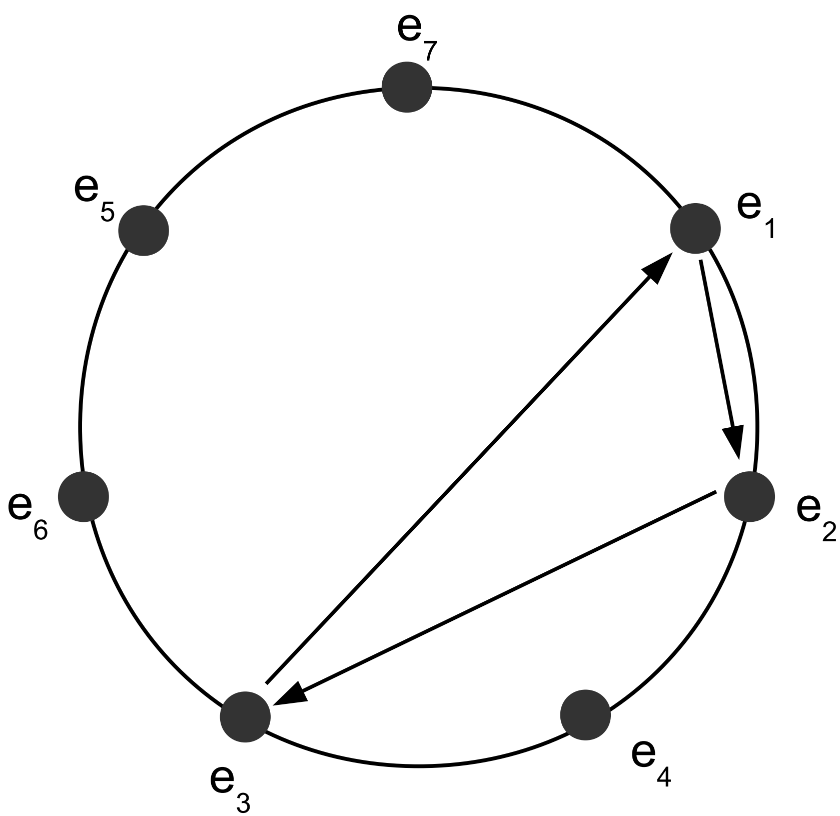
$$\text{Octonions: } 1, e_A \quad A = 1, ..., 7 \qquad e_A e_B = -\delta_{AB} + \epsilon_{ABC} e_C$$

$$\epsilon_{ABC} = 1 for ABC = 123, 516, 624, 435, 471, 673, 572$$



$$
\begin{aligned}
e_1\ e_2 &= e_3 \\
e_2\ e_1 &= -e_3 \\
[e_1, e_2] &= 2e_3
\end{aligned}
$$

Gürsey diagram



$$[x, y, z] = (xy)z - x(yz)$$

$$
\begin{aligned}
e_1\ e_2 &= e_3 \\
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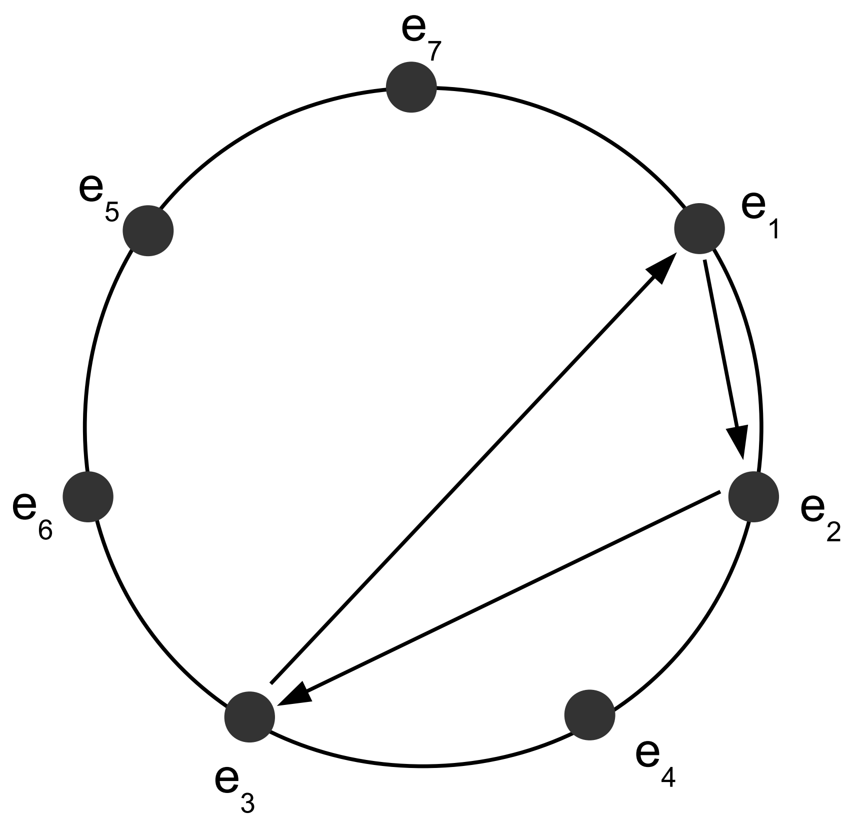
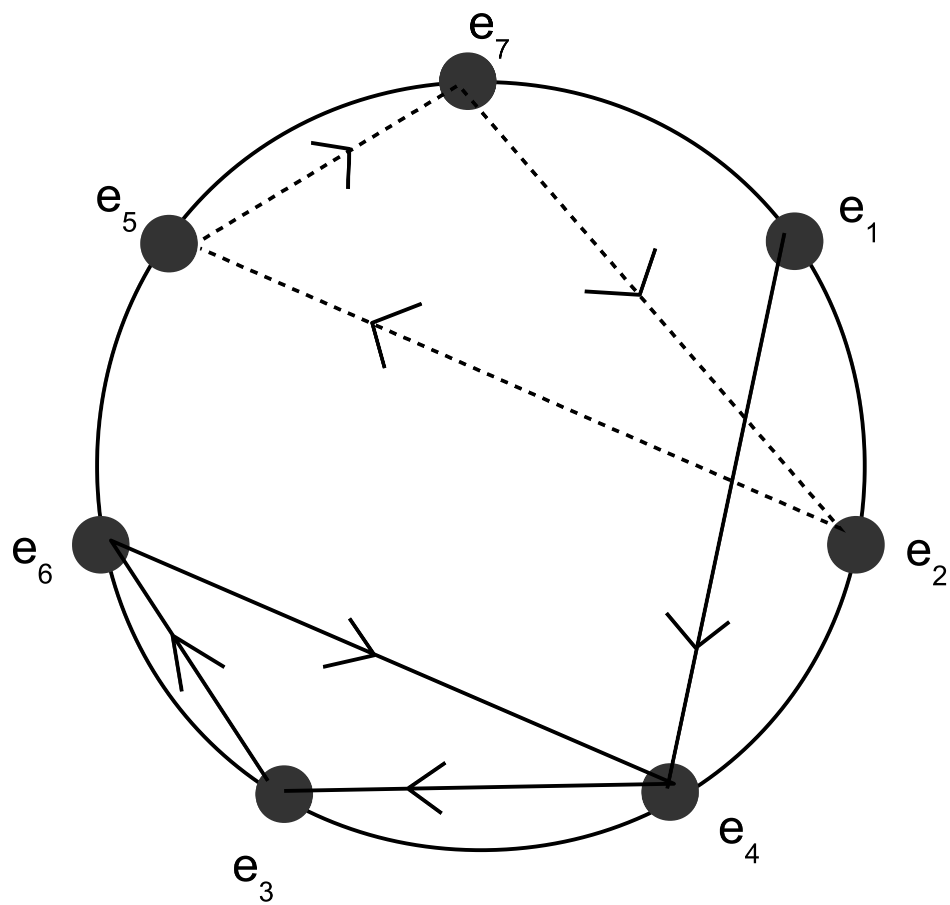
$$



$$[x, y, z] = (xy)z - x(yz)$$

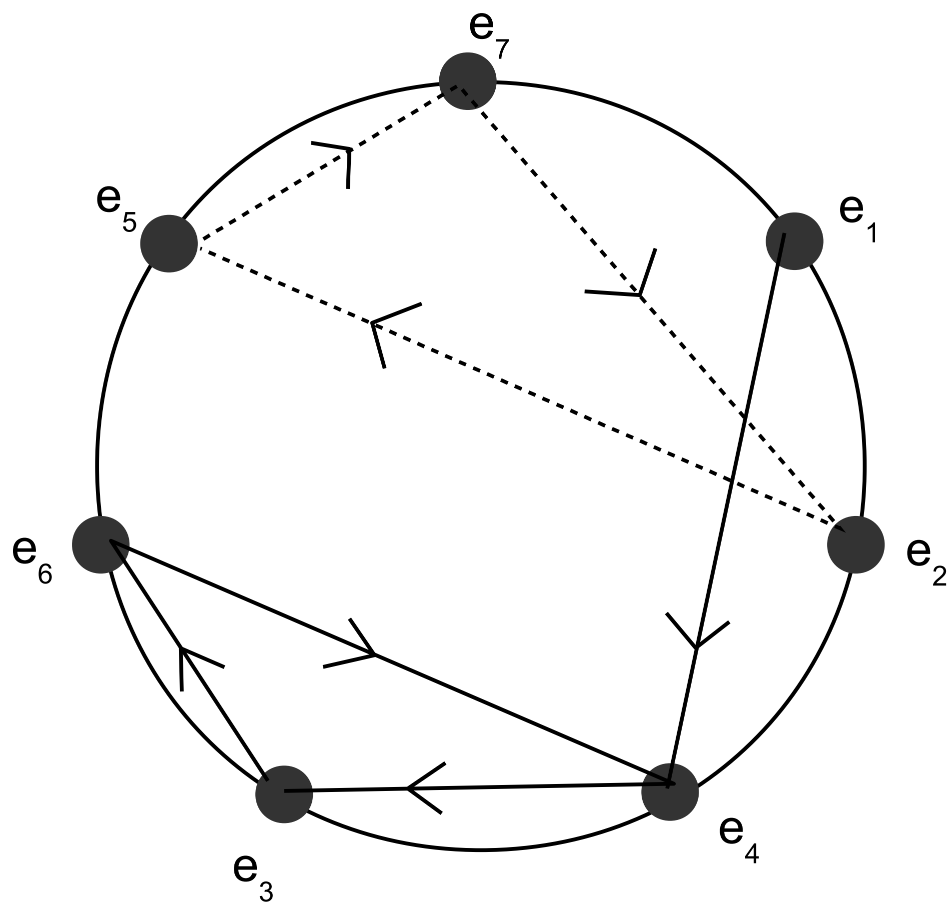
$$[e_4, e_6, e_5] = (e_4 e_6)e_5 - e_4(e_6 e_5) = 2e_7$$

Completion of Gürsey diagram



$$\frac{1}{2}[e_5, e_7] = e_2$$

$$\frac{1}{2}[e_7, e_1] = e_4$$



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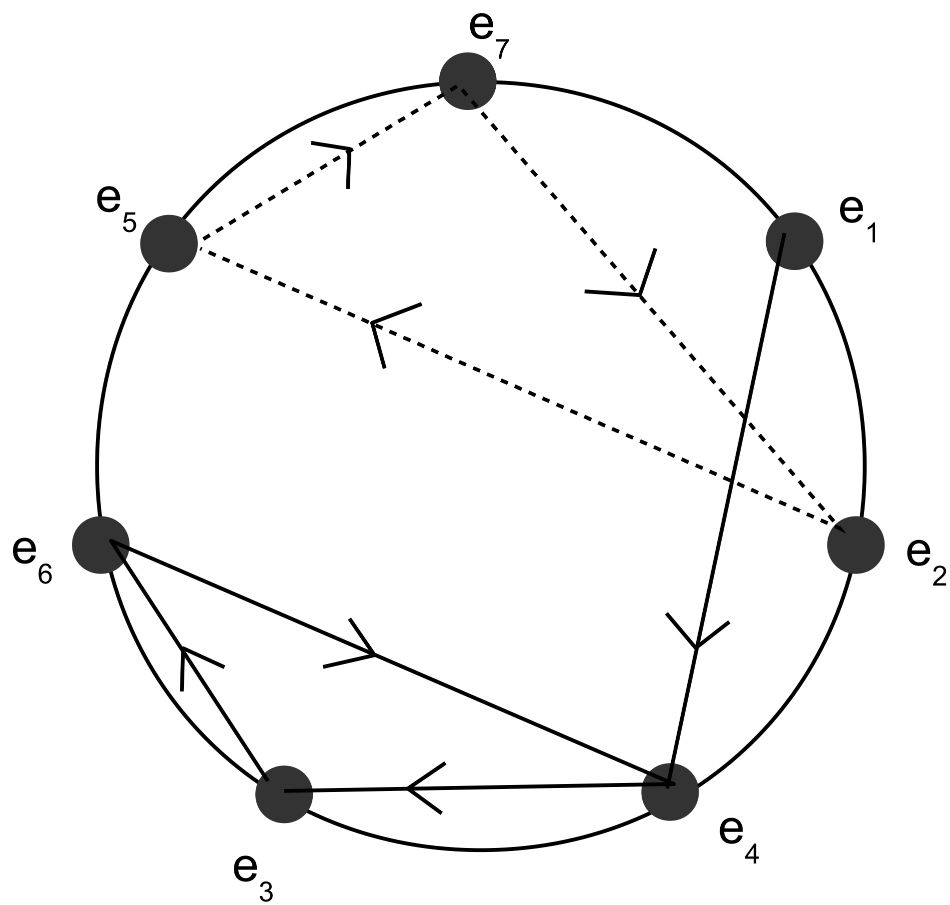

$$\frac{1}{2}[e_1, e_4, e_3] = e_6$$

$$\frac{1}{2}[e_2, e_3, e_6] = e_5$$

$$[x, y, z] = [y, z, x] = [z, x, y]$$

$$[x, y, z] = -[y, x, z] = -[x, z, y] = -[z, y, x]$$

Define a 4-index object $\psi_{\alpha\beta\mu\nu}$ related to the associator as

$$[e_\alpha, e_\beta, e_\mu] = 2\psi_{\alpha\beta\mu\nu} e_\nu$$

$\psi_{\alpha\beta\mu\nu} = 1$ for combinations $1346, 2635, 4567, 3751, 6172, 5214, 7423$

Duality property between $\epsilon_{\lambda\sigma\rho}$ and $\psi_{\alpha\beta\mu\nu}$ in R^7 is best seen in the following construction:

$$\left\{\begin{array}{ccccccc} 5 & 7 & 1 & 2 & 4 & 3 & 6 \\ 7 & 1 & 2 & 4 & 3 & 6 & 5 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \end{array}\right\} = \epsilon_{\lambda\sigma\rho}$$

$$\left\{\begin{array}{ccccccc} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 4 & 3 & 6 & 5 & 7 & 1 & 2 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \\ 6 & 5 & 7 & 1 & 2 & 4 & 3 \end{array}\right\} = \psi_{\alpha\beta\mu\nu}$$

SPLIT OCTONION ALGEBRA

One can form a split Cayley algebra over the field of complex numbers with basis:

$$u_1 = \frac{1}{2}(e_1 + ie_4) \qquad u_1^* = \frac{1}{2}(e_1 - ie_4)$$

$$u_2 = \frac{1}{2}(e_2 + ie_5) \qquad u_2^* = \frac{1}{2}(e_2 - ie_5)$$

$$u_3 = \frac{1}{2}(e_3 + ie_6) \qquad u_3^* = \frac{1}{2}(e_3 - ie_6)$$

$$u_0 = \frac{1}{2}(1 + ie_7) \qquad u_0^* = \frac{1}{2}(1 - ie_7)$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units $e_\alpha, \alpha = 1, 2, \ldots, 7$ fall into its 7-dim representation.

Under the $SU(3)^c$ subgroup of the G_2 that leaves e_7 invariant, u_0 and u_0^* transform like singlets while u_j and u_j^* transform like a triplet and an antitriplet respectively. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner:

$$u_0^2 = u_0 \qquad u_0 u_0^* = 0$$

$$u_0 u_j = u_j u_0^* = u_j \qquad u_0^* u_j = u_j u_0 = 0$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^*$$

$$u_i u_j^* = -\delta_{ij} u_0$$

To compactify our notation we write:

$$u_0 = \frac{1}{2}(1 + ie_7) \qquad u_0^* = \frac{1}{2}(1 - ie_7)$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}) \qquad u_j^* = \frac{1}{2}(e_j - ie_{j+3}) \quad j = 1, 2, 3$$

$Multiplication Table:$

	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki}u_i^*$	$-\delta_{jk}u_0$
u_j^*	u_j^*	0	$-\delta_{jk}u_o^*$	$\epsilon_{jki}u_i$

Note: u_i and u_j^* behave like fermionic creation and annihilation operators:

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0 \qquad \{u_i, u_j^*\} = -\delta_{ij}$$

Showing the three split units to be Grassmann numbers.
Being non-associative they give rise to an exceptional Grassmann algebra.

DYNAMICAL SUPERSYMMETRY

Under the color group $SU(3)^c$

$$q\bar{q}: \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

$$qq: \quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$$

Under the spin-flavor group $SU_{sf}(6)$

$$q\bar{q}: \quad \mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{1} \oplus \mathbf{35}$$

$$qq: \quad \mathbf{6} \otimes \mathbf{6} = \mathbf{15} \oplus \mathbf{21}$$

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qq : $\quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6} \qquad u_j u_k = \epsilon_{jki} u_i^*$

Under the spin-flavor group $SU_{sf}(6)$

$q\bar{q}$: $\quad \mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{1} \oplus \mathbf{35}$

qq : $\quad \mathbf{6} \otimes \mathbf{6} = \mathbf{15} \oplus \mathbf{21}$

If one re-writes qqq baryon as qD, where D is a diquark, the quantum numbers of D are:

for color, $\bar{3}$, since when combined with q must give a color singlet;
for spin-flavor, 21, since when combined with color must give antisymmetric wavefunctions.

But the quantum numbers of $\bar{q}$ are:

for color, $\bar{3}$,
and for spin-flavor, $\bar{6}$.

Thus $\bar{q}$ and D have the same color quantum numbers (color forces can not distinguish between $\bar{q}$ and D).

- Dimensions of Internal Degrees of Freedom of Quarks & Diquarks

	$\mathbf{SU_f(3)}$	$\mathbf{SU_s(2)}$	**dim**
q	$\square$	s=1/2	$3 \times 2 = 6$
D	$\square\square$	$s = 1$	$6 \times 3 = 18$
	$\begin{array}{c}\square\\\square\end{array}$	$s = 0$	$3 \times 1 = 3$

Thus there is an approximate dynamic supersymmetry in hadrons with supersymmetric partners

$$\psi = \begin{pmatrix} \bar{q} \\ D \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} q \\ \bar{D} \end{pmatrix}$$

All hadrons can be obtained by combining ψ and $\bar{\psi}$:
mesons are $q\bar{q}$, baryons are qD,
antibaryons are $\bar{q}\bar{D}$, and exotic mesons are $D\bar{D}$.

Corresponding supersymmetry is $SU(6/21)$.

The confining energy associated with the Bohr radius for the bound state is obtained from the linear confining potential $S(r) = br$, so that the effective masses of the constituents become:

$$M_1 = m_1 + \frac{1}{2}S_0 \qquad M_2 = m_2 + \frac{1}{2}S_0 \qquad (S_0 = br_0)$$

For a meson m_1 and m_2 are the current quark masses while M_1 and M_2 can be interpreted as the constituent quark masses. Note that even in the case of vanishing quark masses associated with a perfect chiral symmetry, confinement results in non-zero constituent masses that spontaneously break the $SU(2) \times SU(2)$ symmetry of u, d quarks.

Simplified spin free Hamiltonian involving only the scalar potential:

$$H^2 = 4[(m + \frac{1}{2}br)^2 + P_r^2 + \frac{\ell(\ell+1)}{r^2}]$$

$$P_r^2 = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}$$

$$Potential\ model\ gives:$$

$$\frac{9}{8}(m_\rho^2 - m_\pi^2) = m_\Delta^2 - m_N^2$$

$$with\ an\ accuracy\ of\ 1\%\ of\ the\ experiment.$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \left[\frac{E^2}{4} - \frac{1}{4}b^2(r + \frac{2m}{b})^2 - \frac{\ell(\ell+1)}{r^2}\right]\psi = 0$$

$$\psi = N r^\ell e^{-\frac{1}{4}br^2} F(\alpha = -n_r + 1, \gamma = \ell + \frac{2}{3}, x = \frac{1}{2}br^2) Y_\ell^m(\theta,\phi)$$

$$E^2_{min} = 4\mu^2(\ell + \frac{1}{2})$$

$$
N = \Bigg[\frac{|\alpha|!(2\gamma-1)!(\gamma+|\alpha|-1)!\sqrt{\pi}}{2^\gamma b^\gamma (\gamma-\frac{1}{2})!(\gamma-1)!}
$$

$$
-m \Bigg\{ \sum_{p=0}^{|\alpha|} \frac{2^{\gamma+\frac{1}{2}}(\gamma+p-\frac{1}{2})!}{b^{\gamma+\frac{1}{2}} p!} \left(\frac{|\alpha|!(\frac{1}{2})!}{(|\alpha|-p)!(p-|\alpha|+\frac{1}{2})!} \right)^2
$$

$$
+ \sum_{n=0}^{n_r-1} \frac{(n_r-1)!(\ell+n_r-\frac{1}{2})!(2n)!(\ell+n)!(\ell+2n+1)}{2^{2n}(n!)^2(n_r-n-1)!(\ell+n+\frac{1}{2})!}
$$

$$
\times \sum_{k=n}^{n_\xi-1} \frac{(-1)^{k+|\alpha|+1} 2^{n+\gamma+\frac{1}{2}}(-\frac{1}{2})!(n_\xi-n-1)!(n+k+\gamma-\frac{1}{2})!(n+k+\frac{1}{2})!}{b^{n+\gamma+\frac{1}{2}}(k+\frac{1}{2})!(n_\xi-k-1)!(\ell+k+1)!(n+k-|\alpha|+\frac{1}{2})!} \Bigg\} \Bigg]^{-\frac{1}{2}}
$$

$$E_r^2 = 4b(\ell + 2n_r - \frac{1}{2})$$

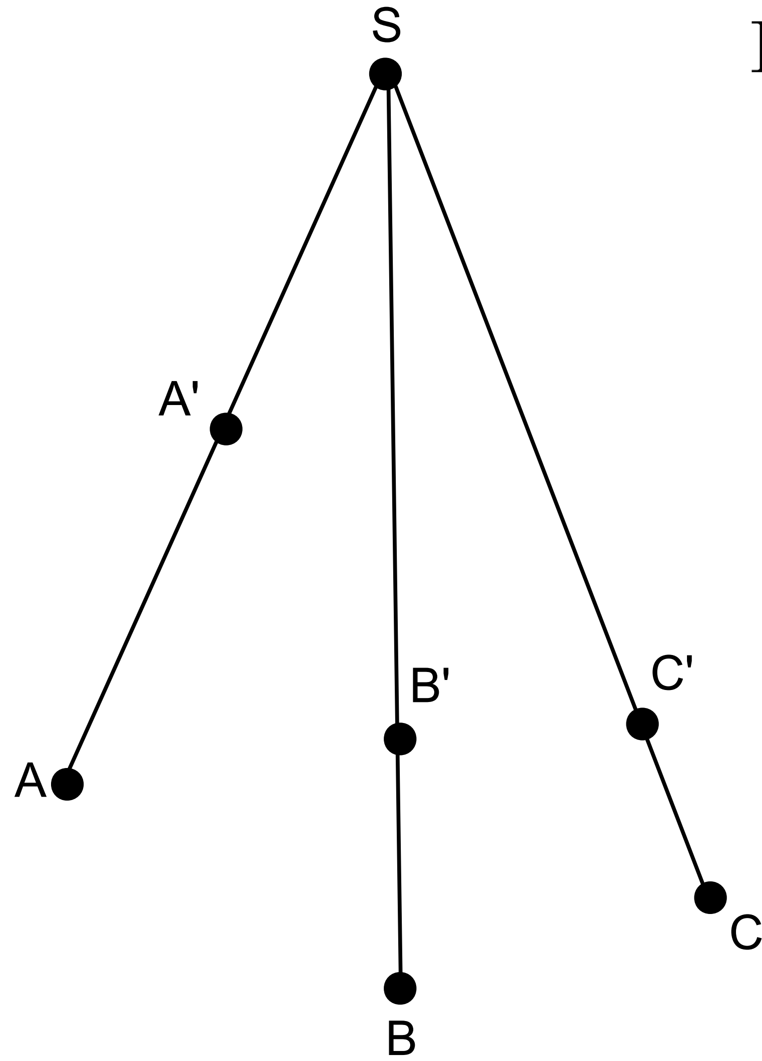
$$E_\xi^2 = 4b(\ell + 2n_\xi + \frac{1}{2})$$

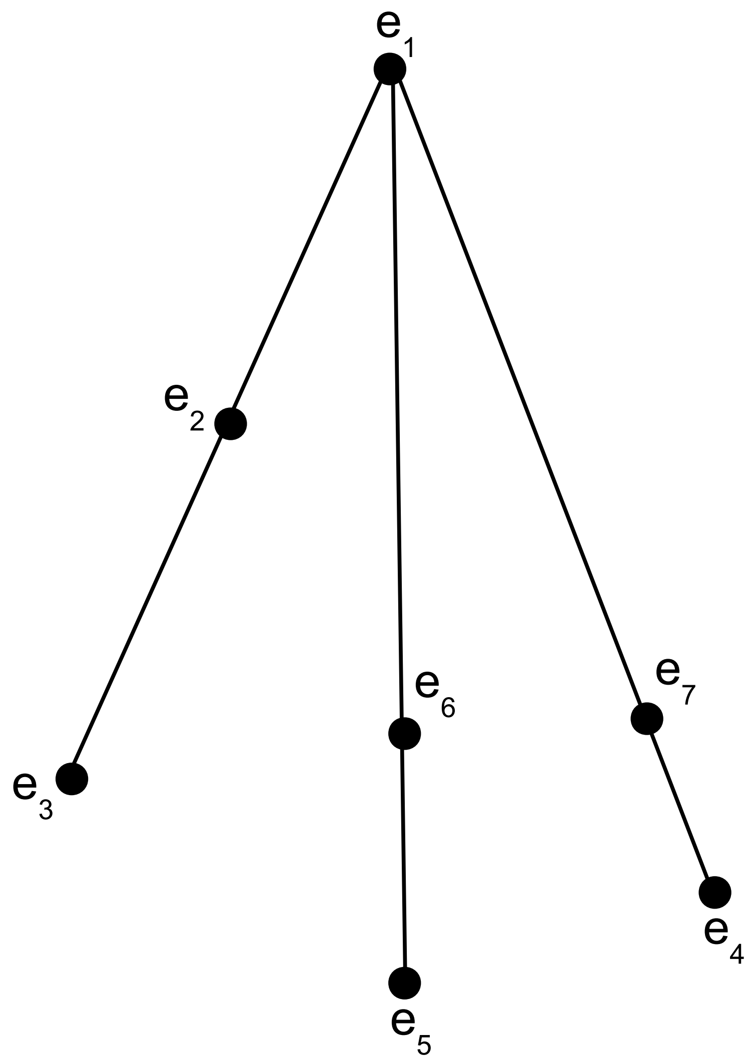
$$n_\xi \geq n_r$$

$RESULTS$

* Parallelism of Regge Trajectories
* Mass formulas $\pi - \rho$, $N - \Delta$ trajectories
* Existence of exotic mesons as $D\bar{D}$ states: $a_0(980)$, $f_0(980)$
* Multiquark states by $q \to \bar{D}$, $\bar{q} \to D$ transform

Desargues' Theorem





e_1
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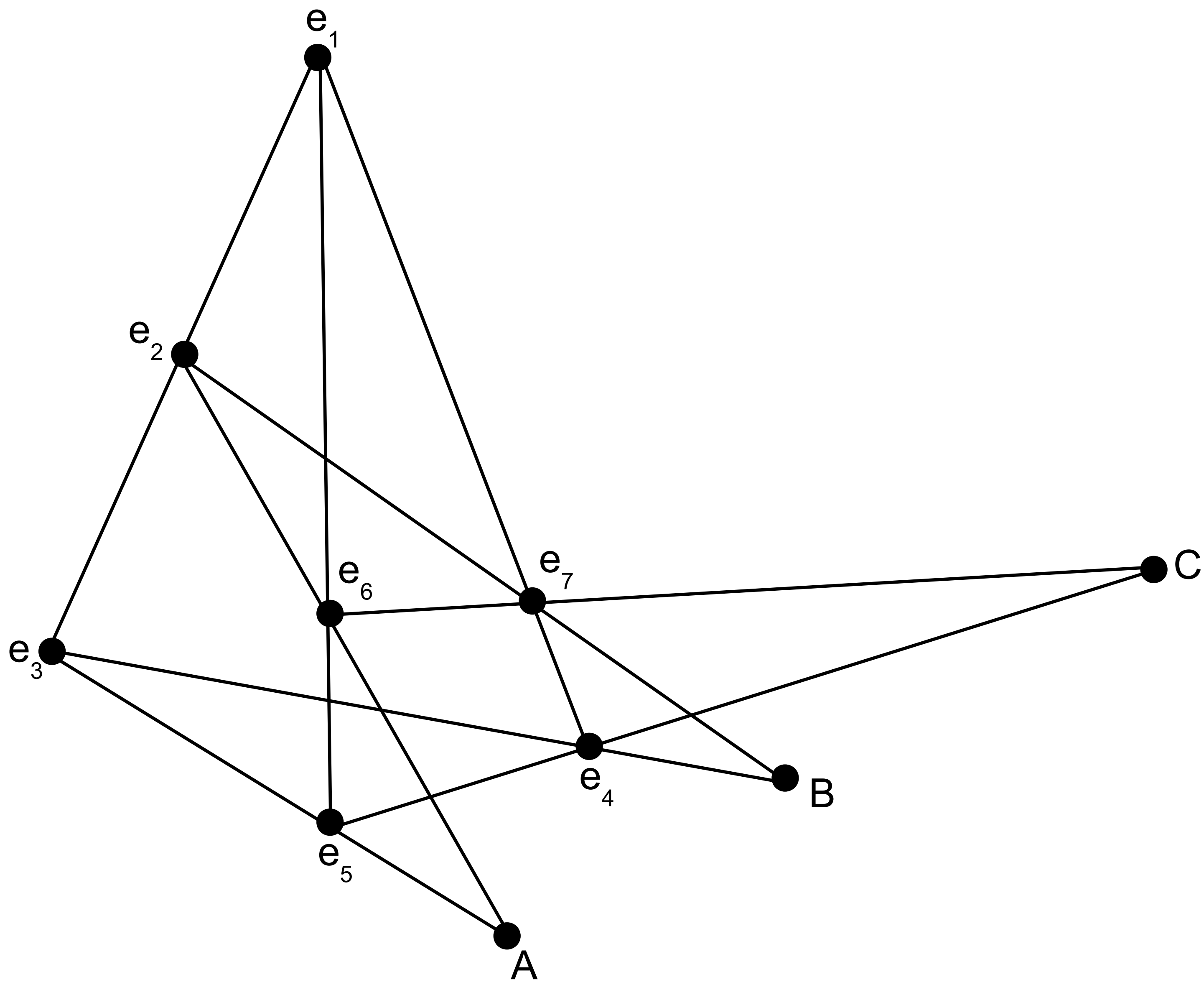
e_3
e_6
e_5
e_7
e_4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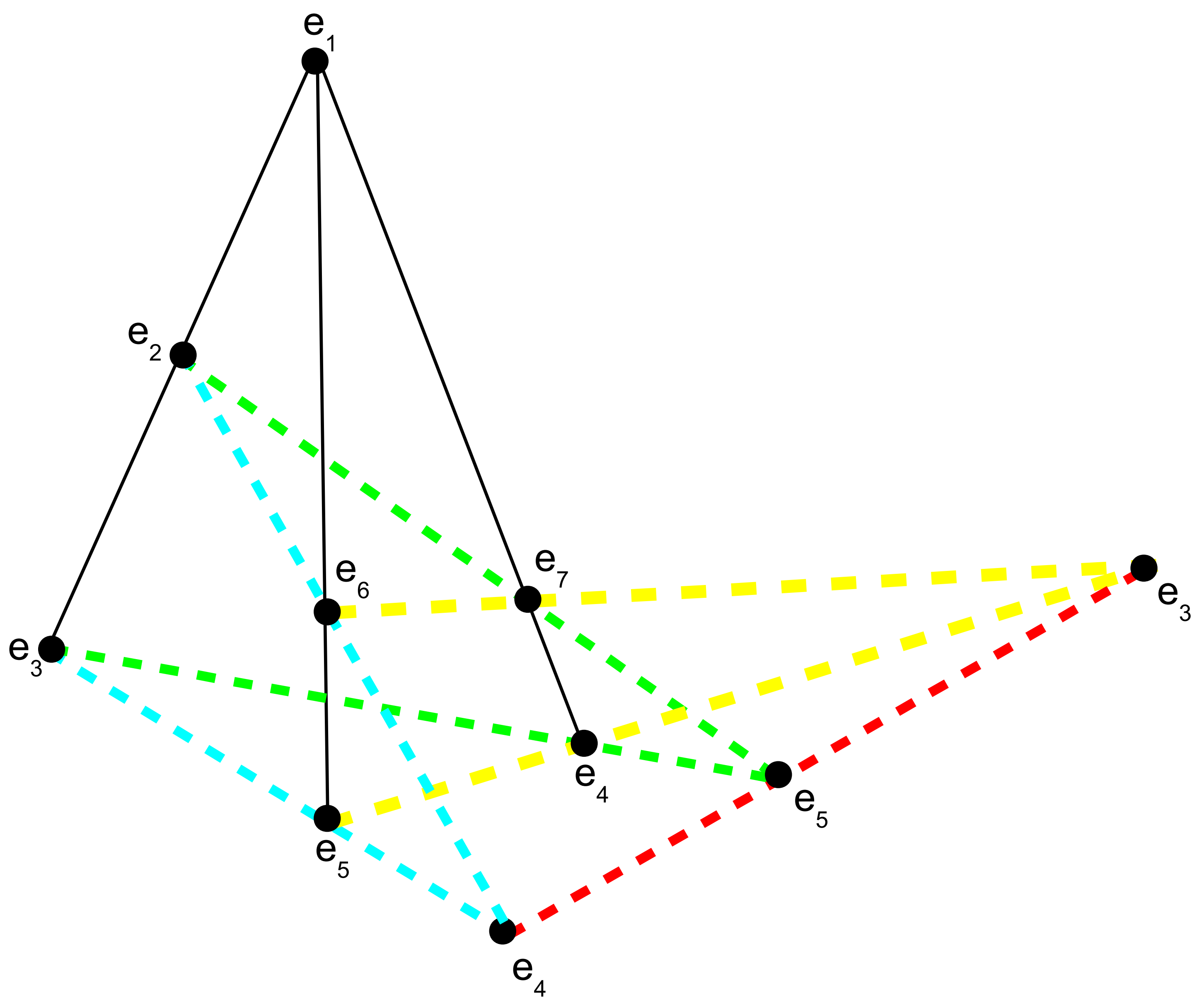
A
B
C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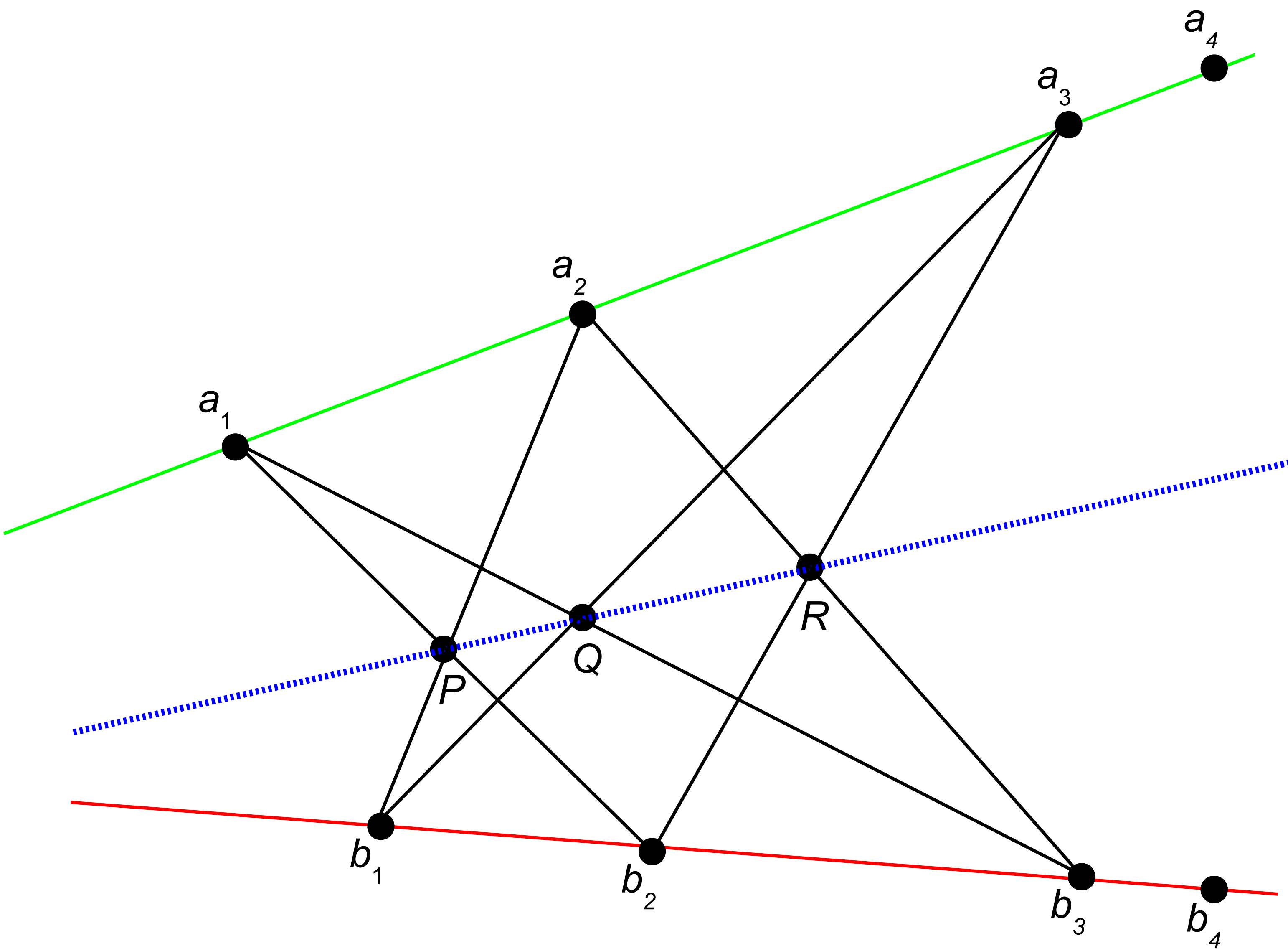
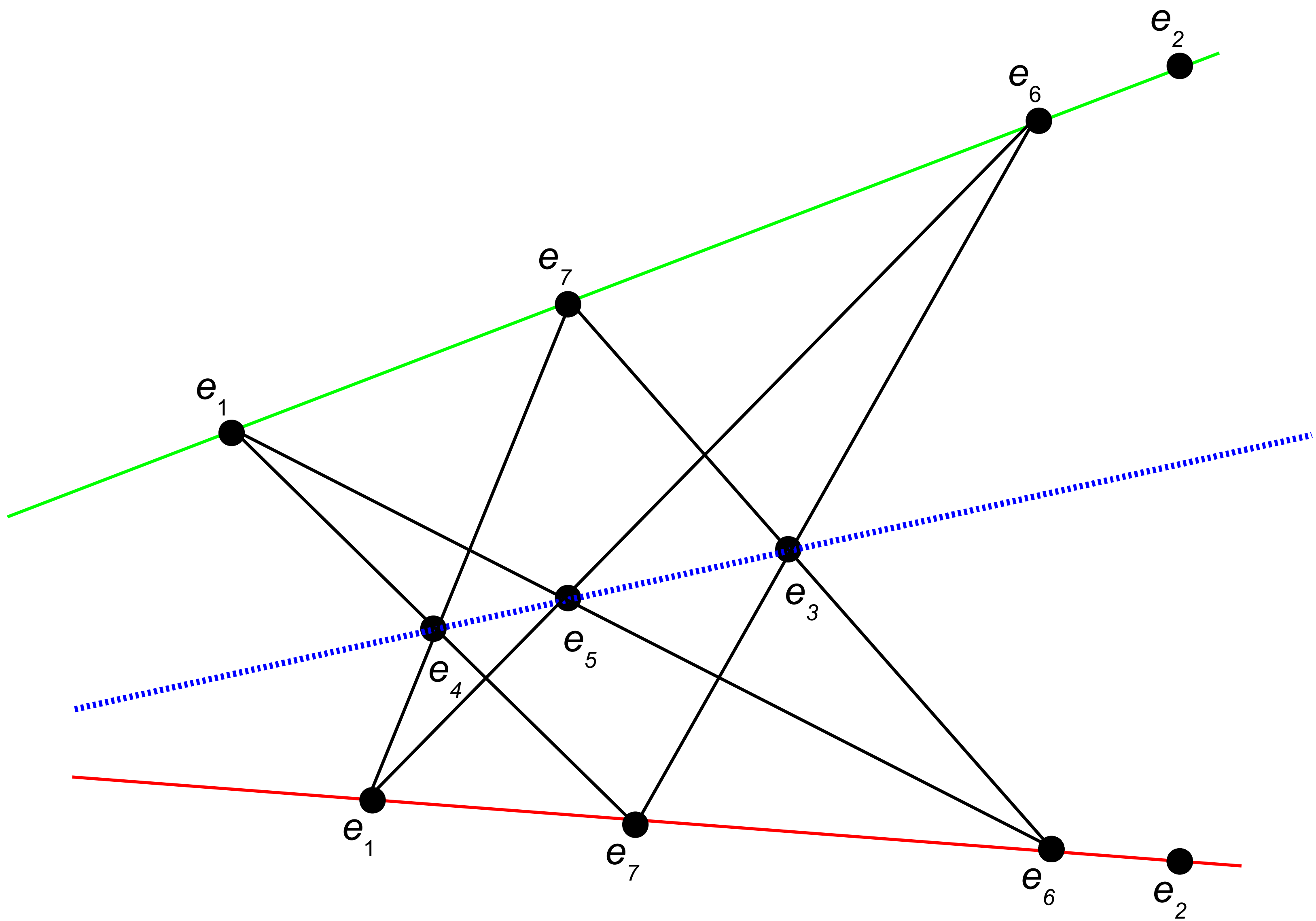
Pappus' Theorem





e2
e6
e7
e1
e3
e5
e4
e1
e7
e6
e2

$\mathbf{Exceptional\,Groups: G_2, F_4, E_6, E_7, E_8}$

$\mathbf{Construction\,of\,the\,root\,lattices\,of\,E_8, E_9 = \hat{E}_8, or\,E_{10}}$

-Conway-Slone lattice associated with discrete Jordan algebras over octonions

-Association between superstring symmetries and lattices generated by discrete Jordan algebras

-Suggest all known superstring theories are related and originated from a more general theory related to Conway-Slone transhyperbolic group