A new concept of spin and orbital momentum operators

Taras Zagoskin

Kharkov Institute of Physics and Technology

Existing concepts of spin and orbital momentum

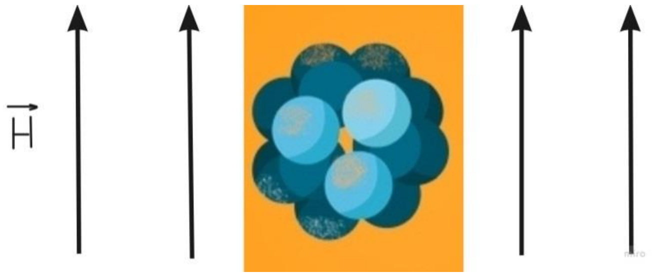
We consider non-relativistic quantum mechanics.

Spin is the intrinsic angular momentum of a particle.

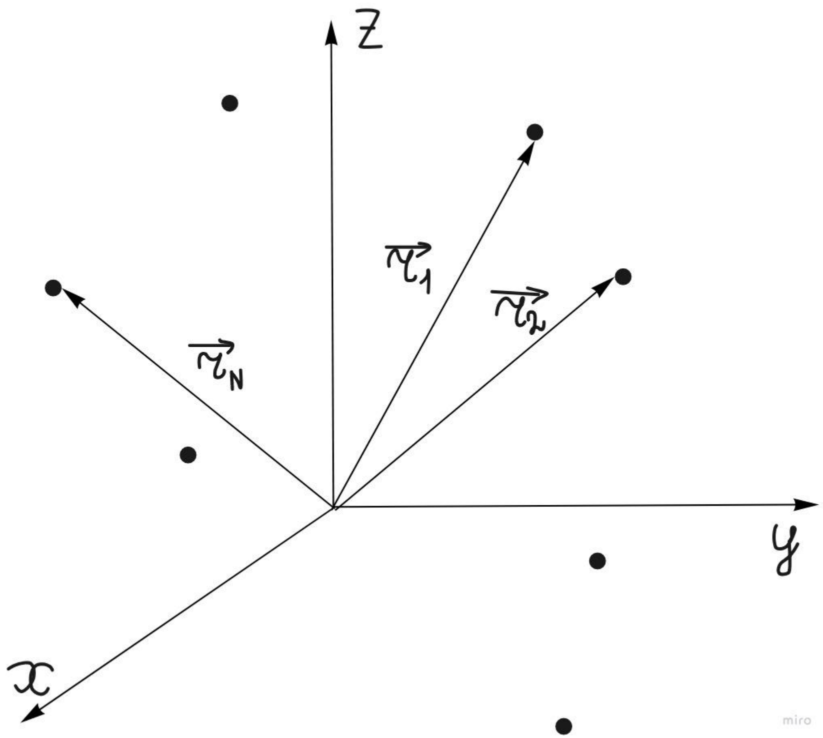
Orbital momentum is its angular momentum connected with its motion in space.

When we will clarify the definitions of spin and orbital momentum,

- we will better understand what they are;
- perhaps we will get new theoretical predictions about processes related to spins and orbital momenta of particles.

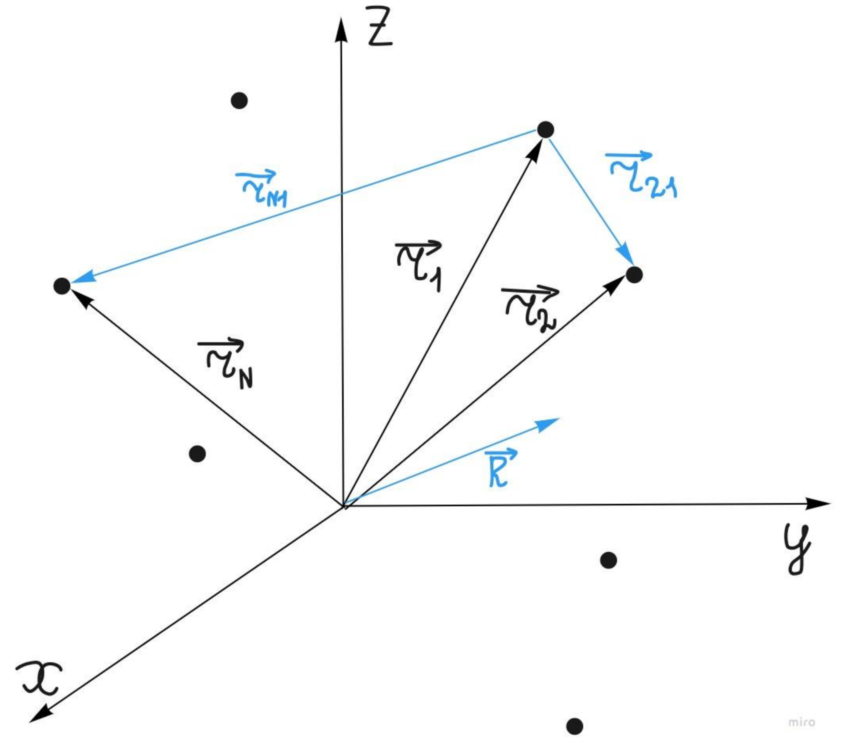


A set of point-like particles



$\widehat{\vec{M}} \equiv -i\hbar \sum_{k=1}^{N} \vec{r}_k \times \frac{\partial}{\partial \vec{r}_k}$ is called the angular momentum operator of the set of particles.

A set of point-like particles



$\vec{r}_{k1} \equiv \vec{r}_k - \vec{r}_1$ are relative position vectors.

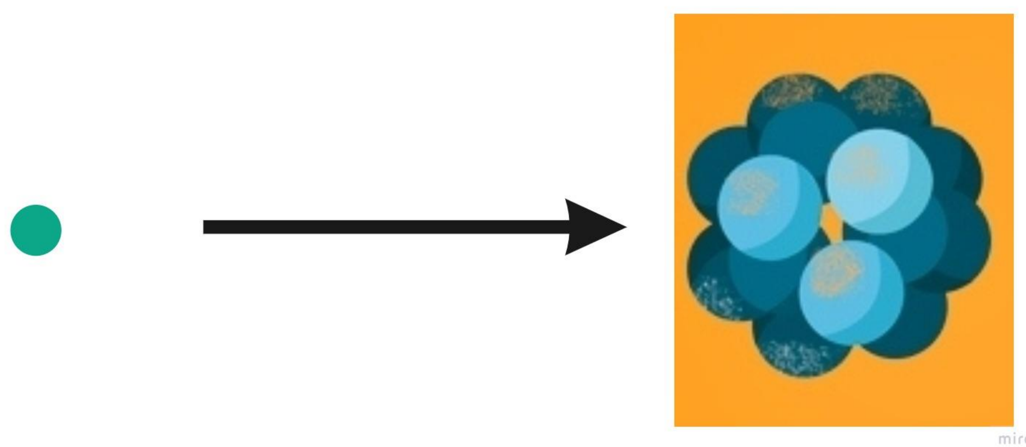
$\vec{R} \equiv \frac{\sum_{k=1}^{N} m_k \vec{r}_k}{\sum_{k=1}^{N} m_k}$ is the position vector of the center of mass.

$$\hat{\vec{M}} \equiv -i\hbar \sum_{k=1}^{N} \vec{r}_k \times \frac{\partial}{\partial \vec{r}_k} = -i\hbar \vec{R} \times \frac{\partial}{\partial \vec{R}} - i\hbar \sum_{k=2}^{N} \vec{r}_{k1} \times \frac{\partial}{\partial \vec{r}_{k1}}$$

Operators of momenta

We introduce the following definitions:

- $\hat{\vec{L}} \equiv -i\hbar\vec{R} \times \frac{\partial}{\partial\vec{R}}$ the operator of orbital momentum of the set of particles 1, 2, …, N;
- $\hat{\vec{S}} \equiv -i\hbar \sum_{k=2}^{N} \vec{r}_{k1} \times \frac{\partial}{\partial\vec{r}_{k1}}$ is the spin operator of this set;
- $\hat{\vec{J}} \equiv \hat{\vec{L}} + \hat{\vec{S}}$ is the total momentum operator.



Commutators

Calculation shows that

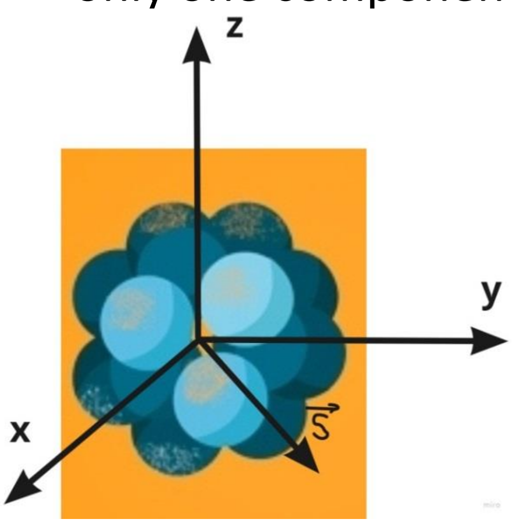
$[\hat{L}_a, \hat{L}_b] = i\varepsilon_{abc}\hat{L}_c,$ $\quad$ $[\hat{L}_a, \hat{S}_b] = 0,$

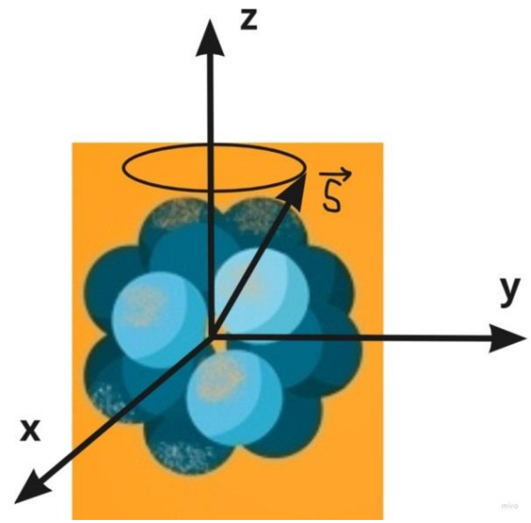
$[\hat{S}_a, \hat{S}_b] = i\varepsilon_{abc}\hat{S}_c,$ $\quad$ $[\hat{J}_a, \hat{J}_b] = i\varepsilon_{abc}\hat{J}_c.$

These commutators are also written in the standard approach of spin and orbital momentum.

That is why in any state of a set of point-like particles for any of the operators $\hat{\vec{S}}$, $\hat{\vec{L}}$, and $\hat{\vec{J}}$

- none of the components has a definite value OR
- only one component has a definite value.





Spins of elementary particles in classical mechanics

We consider a composite elementary particle in the approximation of classical mechanics.

In this approximation it is impossible to correctly take into account the self-interaction of parts of this particle. That is why we can describe some states of an elementary particle in classical mechanics only using the approximation that this particle is point-like, i.e.

$$\vec{r}_{k1} \equiv \vec{r}_k - \vec{r}_1 \approx 0.$$

Spins of elementary particles in classical mechanics

If $\vec{r}_{k1} \approx 0$,

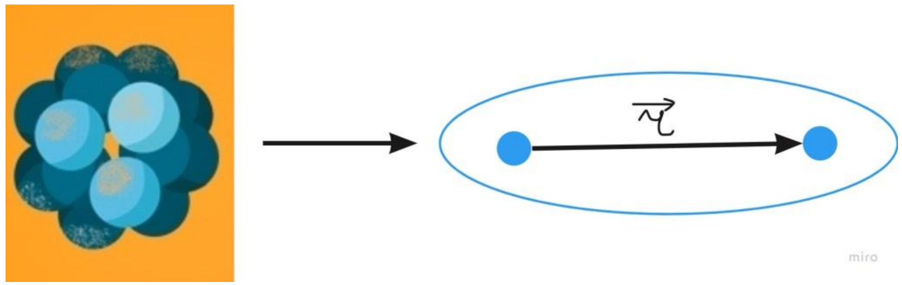
$\hat{\vec{S}} \equiv -i\hbar \sum_{k=2}^{N} \vec{r}_{k1} \times \frac{\partial}{\partial \vec{r}_{k1}} \approx 0$ and

$\hat{\vec{S}}\Psi \approx 0 \cdot \Psi$.

That is why the spin of any elementary particle in the approximation of classical mechanics is zero.

Eigenfunctions and eigenvalues of the operator $\hat{s}^2$

Let us consider a composite particle consisting of two point-like ones.



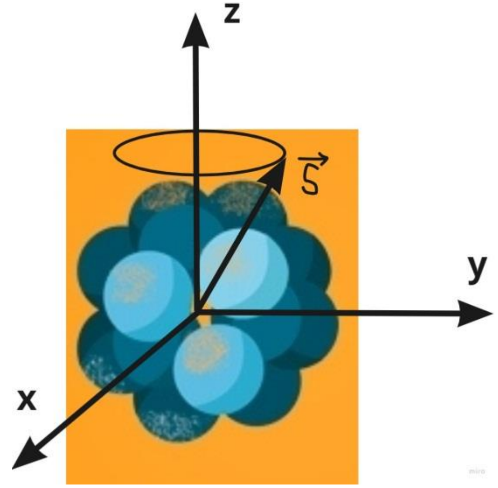
$$\hat{\vec{s}} \equiv -i\vec{r} \times \frac{\partial}{\partial \vec{r}}$$

$$\hat{s}^2\Psi = s(s+1)\Psi \qquad (1)$$

According to experiments, the spins of particles can be integer or half-integer.

- For integer values of s solutions to Eq. (1) are associated Legendre polynomials.
- We have found solutions to this equation for both integer and half-integer values of s.

The “generalized Pauli matrices”



Calculation gives that at $s = \frac{1}{2}$ the matrix of the operator $2\hat{\vec{s}}$ in the basis of eigenfunctions of the operators $\hat{s}^2$ and $\hat{s}_z$ is

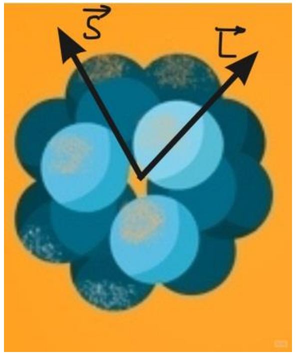
$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \vec{e}_x + \begin{pmatrix} 0 & -iB \\ iB^* & 0 \end{pmatrix} \vec{e}_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{e}_z,$$

where B is a certain complex parameter.

If $B = 1$, the matrices of the operators $2\hat{s}_x$, $2\hat{s}_y$, and $2\hat{s}_z$ coincide with the Pauli martices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conclusions



- We have introduced clear definitions of the spin operator and orbital momentum operator of a set of any number of point-like particles.
- From these definitions, we have derived the following widely used properties of spin and orbital momentum:
  - their commutators;
  - vanishing of the spins of elementary particles in classical mechanics;
  - the Pauli matrices.
- We have obtained spin matrices more general than the Pauli ones.

That is why the suggested concept

- clearly explains what spin and orbital momentum are;
- perhaps will give new theoretical predictions about processes connected with spins and orbital momenta of particles.