Superstatistics and the QCD Critical End Point

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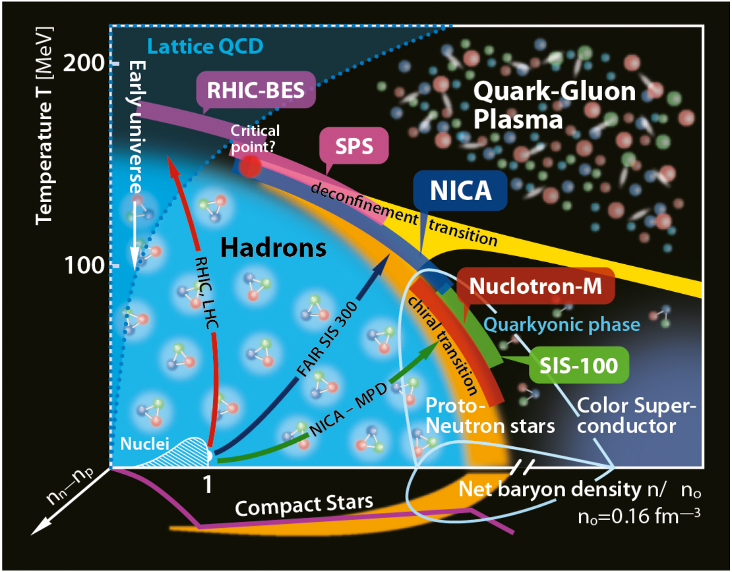
May, 2019

Overview

1. QCD phase diagram
   - Finite temperature and vanishing baryon chemical potential
   - Non-vanishing baryon chemical potential: The sign problem.
2. Superstatistics
   - Average Boltzmann factor
   - Average Partition function
3. QCD phase diagram from chiral symmetry restoration
   - Linear sigma model with quarks
   - Effective potential
4. Results: Critical End Point location
5. Conclusions

QCD phase diagram: current and future experiments



QCD phase diagram, main features:

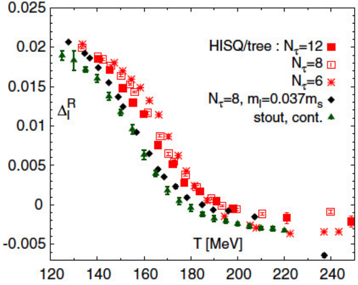
- It is an analytic crossover for $\mu = 0$ (there are no divergences in thermodynamic quantities). There are no symmetries to break. It would be a real phase transition for massless quarks.

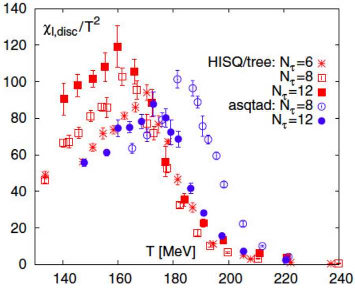
- For $T = 0$ it is a first order phase transition

- The first order phase transition turns into a crossover somewhere in the middle

Light quark condensate $\langle\bar{\psi}\psi\rangle$ from lattice QCD

A. Bazavov *et al.*, Phys. Rev. D **85**, 054503 (2012).





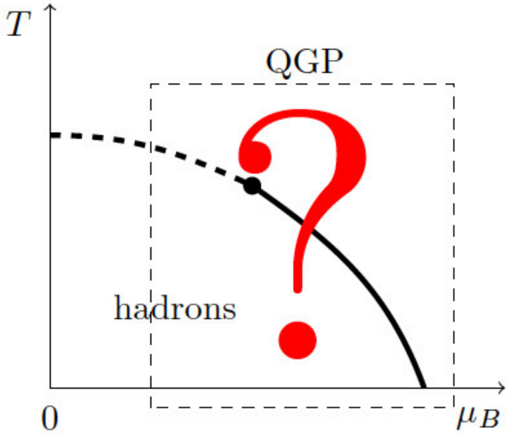
Critical temperatures from lattice QCD for $\mu = 0$

T_c from susceptibility's peak for 2+1 flavors using different kinds of fermion representations. Values show some discrepancies:

- MILC collaboration: $T_c = 169(12)(4)$ MeV.
- BNL-RBC-Bielefeld collaboration: $T_c = 192(7)(4)$ MeV.
- Wuppertal-Budapest collaboration has consistently obtained smaller values, the last being $T_c = 147(2)(3)$ MeV.
- HotQCD collaboration: $T_c = 154(9)$ MeV.

Differences may be attributed to different lattice spacings

For $\mu_B \neq 0$ matters get complicated: Sign problem



The sign problem

- Lattice QCD is affected by the **sign problem**
- The calculation of the partition function produces a fermion determinant.

$$\mathrm{Det}M = \mathrm{Det}(\not{D} + m + \mu\gamma_0)$$

- Consider a complex value for μ. Take the determinant on both sides of the identity

$$\gamma_5(\not{D} + m + \mu\gamma_0)\gamma_5 = (\not{D} + m - \mu^*\gamma_0)^\dagger,$$

we obtain

$$\mathrm{Det}(\not{D} + m + \mu\gamma_0) = [\mathrm{Det}(\not{D} + m - \mu^*\gamma_0)]^*,$$

This shows that **the determinant is not real unless $\mu = 0$ or purely imaginary**

The sign problem

For real μ **it is not possible to carry out the direct sampling on a finite density ensemble by Monte Carlo methods**

- It'd seem that the problem is not so bad since we could naively write

$$\mathrm{Det}M = |\mathrm{Det}M| e^{i\theta}$$

- To compute the thermal average of an observable O we write

$$\langle O \rangle = \frac{\int DUe^{-S_{YM}}\mathrm{Det}M \; O}{\int DUe^{-S_{YM}}\mathrm{Det}M} = \frac{\int DUe^{-S_{YM}}|\mathrm{Det}M|e^{i\theta} \; O}{\int DUe^{-S_{YM}}|\mathrm{Det}M|e^{i\theta}},$$

- S_{YM} is the Yang-Mills action.

The sign problem

- Written in this way, the simulations can be made in terms of the *phase quenched theory* where the measure involves $|\text{Det}M|$ and the thermal average can be written as

$$\langle O \rangle = \frac{\langle O e^{i\theta} \rangle_{\text{pq}}}{\langle e^{i\theta} \rangle_{\text{pq}}}.$$

- The average phase factor (also called the average sign) in the **phase quenched theory** can be written as

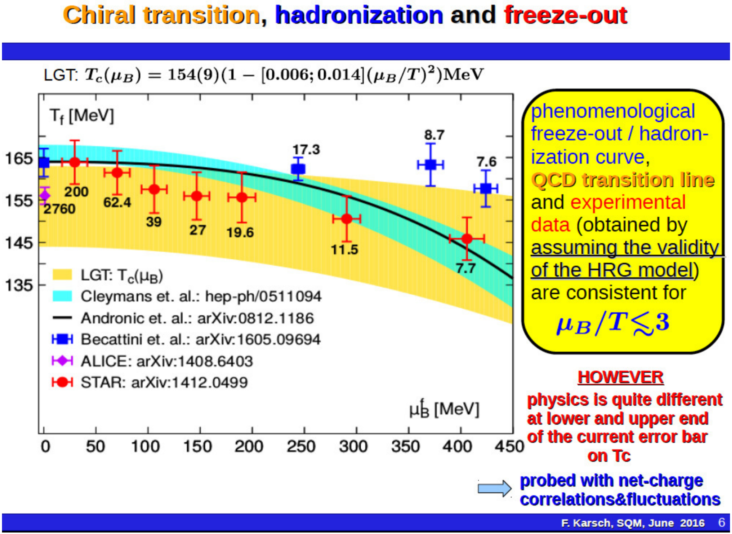
$$\langle e^{i\theta} \rangle_{\text{pq}} = e^{-V(f - f_{pq})/T},$$

where f y f_{pq} are the free energy densities of the full and the phase quenched theories, respectively and V is the 3-dimensional volume.

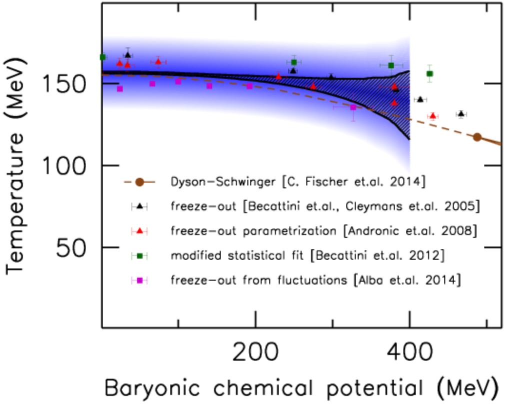
- If $f - f_{pq} \neq 0$, the average phase factor decreaces exponentially when V grows (thermodynamical limit) and/or when T goes to zero.

Under these circumstances the signal/noise ratio worsens.
This is known as the severe sign problem

Chiral transition and freeze out curve

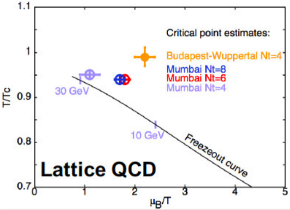


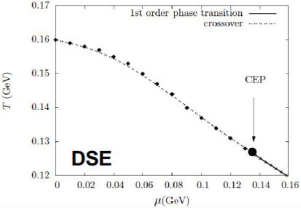
QCD phase diagram from analytic continuation



R. Bellwiede, S. Borsanyi, Z. Fodor, J. Günther, S. D. Katz, C. Ratti, K. K. Szabo, Phys. Lett. B **751**, 559-564 (2015).

The Critical End Point (CEP)





Lattice QCD:

1): Fodor&Katz, JHEP 0404,050 (2004): $(\mu^E_B, T_E) = (360, 162)$ MeV (Reweighting)

2): Gavai&Gupta, NPA 904, 883c (2013) $(\mu^E_B, T_E) = (279, 155)$ MeV (Taylor Expansion)

3): F. Karsch ($\mu^E_B / T_E > 2$, CPOD2016)

DSE:

1): Y. X. Liu, et al., PRD90, 076006 (2014). $(\mu^E_B, T^E) = (372, 129)$ MeV

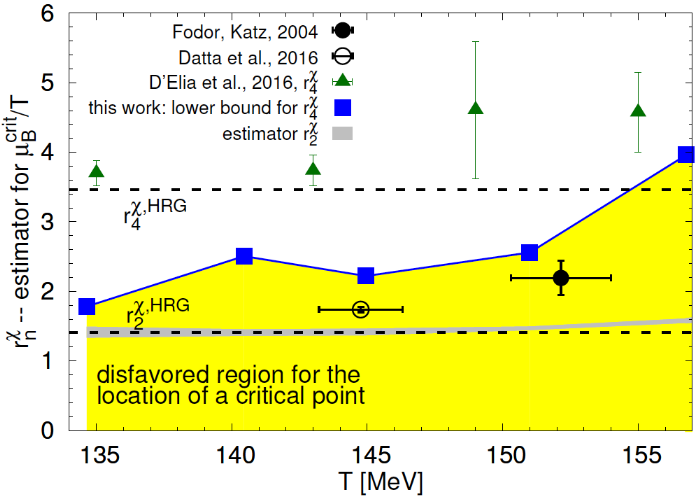
2): Hong-shi Zong et al., JHEP 07, 014 (2014). $(\mu^E_B, T_E) = (405, 127)$ MeV

3): C. S. Fischer et al., PRD90, 034022 (2014). $(\mu^E_B, T^E) = (504, 115)$ MeV

μ^E_B =266 ~ 504 MeV, T_E = 115~162, μ^E_B / T_E =1.8~4.38

Xiaofeng Luo ISMD 2016, Jeju, Korea, Aug. 29-Sept. 2, 2016 3 / 18

CEP location $\mu_B/T > 2$ for 135 MeV $< T <$ 155 MeV



A. Bazavov, *et al.*, Phys. Rev. D **95**, 054504 (2017).

Superstatistics

- Usual thermal description assumes **equilibrium** after some time characterized by values T and μ **taken as common within the whole interaction volume.**
- System evolution subsequently described by time evolution of the temperature down to kinetic freeze-out, where particle spectra are established.
- This picture rests on **two ingredients: the validity of Gibbs-Boltzmann statistics and a system's adiabatic evolution.**
- Adiabatic evolution can perhaps be safely assumed, **however Gibbs-Boltzmann statistics can be applied only to systems long after the relaxation time has elapsed and randomization has been achieved within the system volume.**

Superstatistics

- In the case of a HIC, the reaction starts off from nucleon-nucleon interactions.
- If thermalization is achieved, it seems natural to assume that this starts off in each of the interacting nucleon pair subsystems, and later spreads to the entire volume.
- In this scenario T and μ **within each subsystem may not be the same for other subsystems.**
- A superposition of statistics, one in the usual Gibbs-Boltzmann sense for particles in each subsystem, and another one, for the probability to find particular values for T and μ for different subsystem, seems appropriate.
- **This is described by the so-called superstatistics scenario.**

Average Boltzmann factor

- To implement the scenario, one defines an averaged Boltzmann factor

$$B(\hat{H}) = \int_0^\infty f\ (\beta) e^{-\beta \hat{H}} d\beta,$$

- $f\ (\beta)$ is the probability distribution of β.
- The partition function then becomes

$$\begin{aligned} Z &= \mathrm{Tr}[B(\hat{H})] \\ &= \int_0^\infty B(E) dE, \end{aligned}$$

- The last equality holds for a suitably chosen set of Hamiltonian eigenstates.

Two possible averaging procedures

- Superstatistics **Type A:** compute partition function as the trace of a modified Boltzmann factor coming from first averaging the possible temperature values
- Superstatistics **Type B:** compute first the trace of the Boltzmann factor for each subsystem and then average over the different subsystems temperatures
- Case B can be easily reduced to case A by replacing the distribution $f(\beta)$ by a new distribution $\widetilde{f}(\beta) = CZ^{-1}(\beta)f(\beta)$, where C is a suitable normalization constant.
- Since the normalization factor in the second case depends on β, one can expect a different result when working with the same $f(E)$.
- Nevertheless, different $f(E)$'s lead to similar entropic factors when expanded to first order in $1/N$.

Average Boltzmann factor

- A possible choice to distribute the random variable β is the χ^2 distribution

$$f\,(\beta) = \frac{1}{\Gamma(N/2)}\left(\frac{N}{2\beta_0}\right)^{N/2}\beta^{N/2-1}e^{-N\beta/2\beta_0},$$

- Γ is the Gamma function, N is the number of subsystems

$$\beta_0 \equiv \int_0^\infty \beta f\,(\beta)d\beta = \langle\beta\rangle, \tag{1}$$

- The χ^2 distribution emerges for a variable β made out from the sum of positive definite random variables X_i each of which is Gaussian distributed

$$\beta = \sum_{i=1}^{N} X_i^2, \tag{2}$$

Average Boltzmann factor

- The variance is given by

$$\langle \beta^2 \rangle - \beta_0^2 = \frac{2}{N}\beta_0^2.$$

- Given that β is a positive-definite quantity, thinking of it as being the sum of positive-definite random variables is an adequate model. Notice however that these variables do not necessarily correspond to the inverse temperature in each of the subsystems.
- However, since the use of the χ^2 distribution allows for an analytical treatment, one can take this as the distribution to model the possible values of β.

Average Boltzmann factor

- The **effective Boltzmann factor** is given by

$$B(\hat{H}) = (1 + \frac{2}{N}\beta_0 \hat{H})^{-\frac{N}{2}}.$$

- Notice that in the limit when $N \to \infty$ the effective Boltzmann factor becomes the ordinary one. For large but finite N one can expand as

$$\begin{aligned} B(\hat{H}) &= \left[1 + \frac{1}{2}\left(\frac{2}{N}\right)\beta_0^2 \hat{H}^2 - \frac{1}{3}\left(\frac{2}{N}\right)^2 \beta_0^3 \hat{H}^3 + \cdots\right] \\ &\times \quad e^{-\beta_0 \hat{H}} \end{aligned}$$

- Working up to first order in $1/N$

$$B(\hat{H}) = \left[1 + \frac{\beta_0^2}{N}\left(\frac{\partial}{\partial \beta_0}\right)^2\right] e^{-\beta_0 \hat{H}}.$$

Average Partition function

- Therefore, the partition function to first order in $1/N$ is given by

$$Z = \left[1 + \frac{\beta_0^2}{N}\left(\frac{\partial}{\partial\beta_0}\right)^2\right] Z_0 \tag{3}$$

- with

$$Z_0 = e^{-\mathtt{V}\beta_0 V^{\text{eff}}}, \tag{4}$$

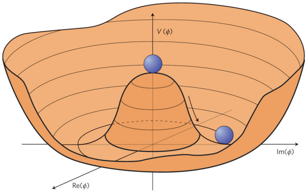
- where $\mathtt{V}$ and V^{eff} are the system's volume and effective potential, respectively.

Model QCD: Linear sigma model

- Effective QCD models (linear sigma model with quarks)

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{a^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 \\
&+ i\bar{\psi}\gamma^\mu \partial_\mu \psi - g\bar{\psi}(\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi})\psi,
\end{aligned}
$$

$$
\begin{aligned}
\sigma &\rightarrow \sigma + v, \\
m_\sigma^2 &= \frac{3}{4}\lambda v^2 - a^2, \\
m_\pi^2 &= \frac{1}{4}\lambda v^2 - a^2 \\
m_f &= gv \\
v_0 &= \frac{2a}{\sqrt{\lambda}}
\end{aligned}
$$



Three level potential (vacuum stability)

$$V^{\text{tree}}(v) = -\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4$$

$$v_0 = \sqrt{\frac{a^2}{\lambda}},$$

$$V^{\text{tree}} = -\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4 \to -\frac{(a^2 + \delta a^2)}{2}v^2 + \frac{(\lambda + \delta\lambda)}{4}v^4.$$

δa^2 and $\delta\lambda$ constants to be determined from the properties of the phase transitions at $(\mu_B = 0, T^c(\mu_B = 0))$ and $(\mu_B^c(T = 0), T = 0)$.

M. E. Carrington, Phys. Rev. D **45**, 2933

One-loop boson and fermion effective potential

$$V^{(1)\mathrm{b}}(v, T) = T \sum_n \int \frac{d^3k}{(2\pi)^3} \ln D(\omega_n, \vec{k})^{1/2},$$

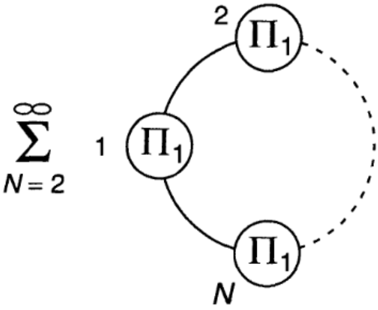
$$D(\omega_n, \vec{k}) = \frac{1}{\omega_n^2 + k^2 + m_b^2},$$

$$V^{(1)\mathrm{f}}(v, T, \mu_q) = -T \sum_n \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}[\ln S(\tilde{\omega}_n - i\mu_q, \vec{k})^{-1}],$$

$$S(\tilde{\omega}_n, \vec{k}) = \frac{1}{\gamma_0 \tilde{\omega}_n + \not{k} + m_f}.$$

$\omega_n = 2n\pi T$ and $\tilde{\omega}_n = (2n+1)\pi T$ are
the boson and fermion Matsubara frequencies

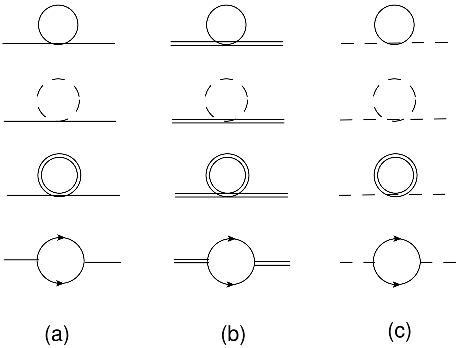
Ring-diagrams effective potential



$$V^{\text{Ring}}(v, T, \mu_q) = \frac{T}{2}\sum_n \int \frac{d^3k}{(2\pi)^3} \times \ln[1 + \Pi(m_b, T, \mu_q) D(\omega_n, \vec{k})]$$

$\Pi(m_b, T, \mu_q)$ is the boson's self-energy.

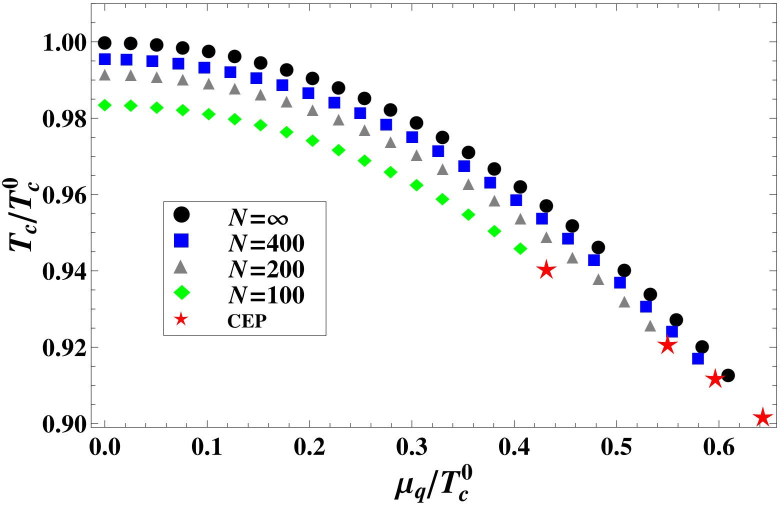
Diagrams contributing to bosons' self-energies



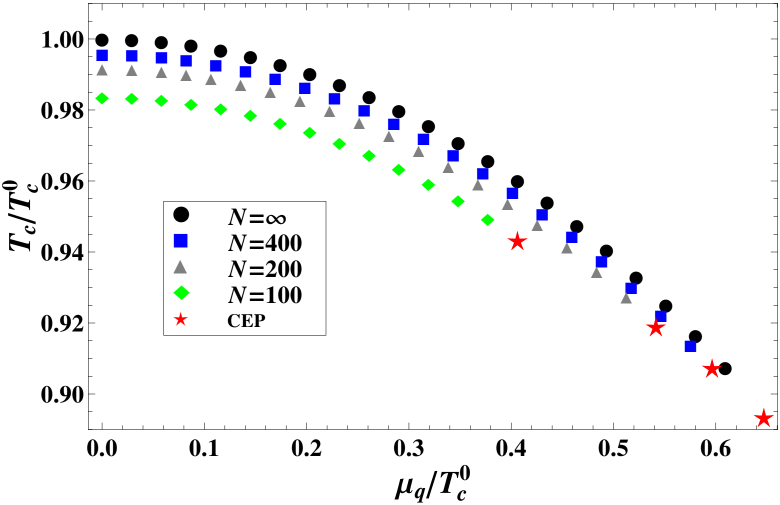
$$\Pi(T,\mu_q) = \quad -N_f N_c g^2 \frac{T^2}{\pi^2}[\mathrm{Li}_2(-e^{\mu_q/T}) + \mathrm{Li}_2(-e^{-\mu_q/T})] + \frac{\lambda T^2}{2}.$$

$$
\begin{aligned}
V^{\rm eff} &= -\frac{(a^2+\delta a^2)}{2}v^2 + \frac{(\lambda+\delta\lambda)}{4}v^4 \\
&+ \sum_{b=\sigma,\bar{\pi}} \Big\{ -\frac{m_b^4}{64\pi^2}\Big[\ln\Big(\frac{a^2}{4\pi T^2}\Big) - \gamma_E + \frac{1}{2}\Big] \\
&- \frac{\pi^2 T^4}{90} + \frac{m_b^2 T^2}{24} - \frac{(m_b^2+\Pi(T,\mu_q))^{3/2}T}{12\pi}\Big\} \\
&+ \sum_{f=u,d} \Big\{ \frac{m_f^4}{16\pi^2}\Big[\ln\Big(\frac{a^2}{4\pi T^2}\Big) - \gamma_E + \frac{1}{2} \\
&- \psi^0\Big(\frac{1}{2}+\frac{\mathrm{i}\mu_q}{2\pi T}\Big) - \psi^0\Big(\frac{1}{2}-\frac{\mathrm{i}\mu_q}{2\pi T}\Big)\Big] \\
&- 8m_f^2T^2\Big[\mathrm{Li}_2(-e^{\mu_q/T}) + \mathrm{Li}_2(-e^{-\mu_q/T})\Big] \\
&+ 32T^4\Big[\mathrm{Li}_4(-e^{\mu_q/T}) + \mathrm{Li}_4(-e^{-\mu_q/T})\Big]\Big\}
\end{aligned}
$$

Effective QCD phase diagram
$a = 133$ MeV, $g = 0.51$, $\lambda = 0.36$,



Effective QCD phase diagram
$a = 133$ MeV, $g = 0.63$, $\lambda = 0.4$



Number of subsystems in a HIC

- To apply these considerations in the context of relativistic heavy-ion collisions, we recall that temperature fluctuations are related to the system's heat capacity by

$$\frac{(1-\xi)}{C_v} = \frac{\langle (T-T_0)^2\rangle}{T_0^2},$$

- $(1-\xi)$ accounts for deviations from the Gaussian distribution for the random variable T.
- Fluctuations in T can be written in terms of fluctuations in β as

$$\begin{aligned}\frac{\langle (T-T_0)^2\rangle}{T_0^2} &= \frac{\beta_0^2-\langle\beta^2\rangle}{\langle\beta^2\rangle}\\ &= \frac{\left(\frac{\beta_0^2}{\langle\beta^2\rangle}\right)^2\langle\beta^2\rangle-\beta_0^2}{\beta_0^2}\end{aligned}$$

Number of subsystems in a HIC

- Recall that

$$\langle \beta^2 \rangle - \beta_0^2 = \frac{2}{N}\beta_0^2.$$

- Thus

$$\begin{aligned} \left(\frac{\beta_0^2}{\langle \beta^2 \rangle}\right)^2 &= \left(\frac{1}{1+2/N}\right)^2 \\ &\simeq 1 - 4/N. \end{aligned}$$

- Therefore, for N finite but large

$$\frac{\langle (T - T_0)^2 \rangle}{T_0^2} \simeq \frac{\langle \beta^2 \rangle - \beta_0^2}{\beta_0^2} = \frac{2}{N},$$

Number of subsystems in a HIC

- This means that the heat capacity is related to the number of subsystems by
$$\frac{(1-\xi)}{C_v} = \frac{2}{N}.$$
- To introduce the specific heat c_v for a HIC, it is natural to divide C_v by the number of participants N_p in the reaction. Therefore
$$\frac{2}{N} = \frac{(1-\xi)}{N_p c_v} = \frac{2}{N} = \frac{(1-N_p/A)}{N_p c_v}.$$
- ξ is estimated as $\xi = N_p/A$, where A is the smallest mass number of the colliding nuclei. This provides the link between N and N_p in a HIC.

Conclusions

- Main goal of future experiments in the field heavy-ion physics is to study QDC at finite baryon density.
- Many challenges. Of particular importance to determine whether there is a CEP.
- If thermalization is local, need to compute average temperature: Superstatistics is an ideal tool.
- Superstatistics combined with effective models are useful tools to gain insight into the possible CEP location when thermalization is partial.
- Linear sigma model is one possibility: Complete exploration requires couplings to bear baryon density and temperature effects.