Institute of Physics
of the Czech
Academy of Sciences

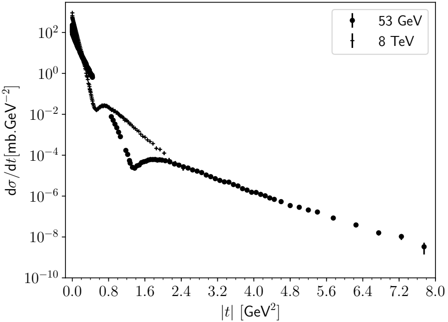
Models of elastic pp scattering at high energies - possibilities, limitations, assumptions and open questions

J. Procházka, V. Kundrát and M. V. Lokajíček

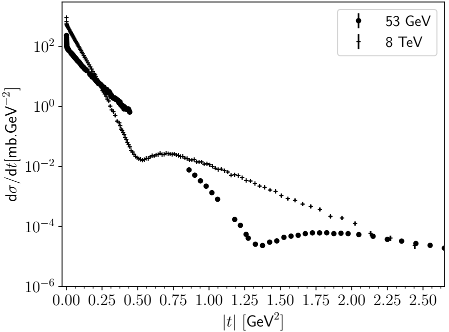
The Czech Academy of Sciences, Institute of Physics

New Trends in High-Energy Physics, Odessa, Ukraine, May 12-18, 2019

Measured elastic pp differential cross section at 52.8 GeV and 8 TeV



(a) $|t|$ values up to 8 GeV^2



(b) $|t|$ values up to $\approx$ 2.5 GeV^2

- 52.8 GeV: ISR data
- 8 TeV: LHC data (TOTEM measurement)
- measured pp $d\sigma/dt$ at both the energies very different, but similar shape:
  - peak at the lowest measured values of $|t|$ (atributed to Coulomb-hadronic interference)
  - followed by nearly exponential t-dependence
  - and dip-bump structure at even higher values of $|t|$

Contemporary situation concerning description of el. scattering

- description and understanding of el. pp scattering is still not fully satisfactory
- contemporary situation summarized recently, e.g., in a strategical document A. Andreazza et al., "What Next: White Paper of the INFN-CSN1", Frascati Phys. Ser. **60**, 1–302 (2015); Section 7.5 - Total, elastic and diffractive cross sections:

  "Several theoretical models have been developed during the last decades to interpret the experimental results. Unfortunately, the perturbative QCD approach cannot be used in this context since most of the processes contributing to the total cross section are characterised by low momentum transfer. Some of the models are still based on Regge theory, while others prefer using optical or eikonal approaches. Moreover, so-called QCD-inspired models are trying to connect the concepts of Pomeron trajectories and proton opacity to the QCD description of elementary interactions between quarks and gluons. At the moment, no model manages to describe qualitatively and quantitatively the large amount of data available; they all have merits and shortcomings. Typically, they successfully describe the experimental results in a certain kinematic range but completely fail in other ones."
- at the moment only the eikonal model approach allows to take into account and study both Coulomb-hadronic interference and dependence of (elastic) particle collisions on impact parameter - more fundamental description than other approaches discussed in the literature

Contemporary descriptions of elastic collisions of charged hadrons

- measured elastic $d\sigma/dt$ of two charged hadrons given by

$$\frac{d\sigma(s,t)}{dt} = \frac{\pi}{sp^2}\left|F^{C+N}(s,t)\right|^2 \quad (1)$$

  - $F^{C+N}(s,t)$ - complete elastic scattering amplitude of Coulomb-hadronic interaction depending on both Coulomb $F^{C}(s,t)$ and hadronic $F^{N}(s,t)$ amplitudes
  - Coulomb interaction is usually assumed to be well known from QED (except from electromagnetic form factors); eq. (1) allows "separation" of Coulomb interaction from data and to study less known elastic hadron (nuclear) scattering (QCD inapplicable)
  - Coulomb-hadronic interference used to constrain t-dependence of phase of $F^{N}(s,t)$

- two approaches for description of elastic collisions of charged hadrons (amplitude $F^{C+N}(s,t)$)
  - West and Yennie (Feynman diagram technique)
  - eikonal model

Coulomb-hadronic interference in the West and Yennie approach

- H. A. Bethe, "Scattering and polarization of protons by nuclei", Ann. Phys. **3**, 190–240 (1958)

$$F^{\mathrm{C+N}}(s,t) = F^{\mathrm{C}}(s,t)\, \mathrm{e}^{\mathrm{i}\alpha\phi(s,t)} + F^{\mathrm{N}}(s,t) \tag{2}$$

- G. B. West and D. R. Yennie, "Coulomb interference in high-energy scattering", Phys. Rev. **172**, 1413–1422 (1968)
  integral formula for relative phase (derived only for "small" values of $|t|$)

$$\alpha\phi(s,t) = \mp\alpha\left[\ln\left(\frac{-t}{s}\right) + \int_{-4p^2}^{0} \frac{\mathrm{d}t'}{|t-t'|}\left(1 - \frac{F^{\mathrm{N}}(s,t')}{F^{\mathrm{N}}(s,t)}\right)\right]. \tag{3}$$

- simplified interference formula of WY (1968)

$$F^{\mathrm{C+N}}_{\mathrm{WY}}(s,t) = \pm\frac{\alpha s}{t} G_1(t) G_2(t)\, \mathrm{e}^{\mathrm{i}\alpha\phi(s,t)} + \frac{\sigma^{\mathrm{tot,N}}(s)}{4\pi} p\sqrt{s}\,(\rho(s)+\mathrm{i})\, \mathrm{e}^{B(s)t/2} \tag{4}$$

  where (see also Locher 1967 [4])

$$\alpha\phi(s,t) = \mp\alpha\left[\ln\left(\frac{-B(s)t}{2}\right) + \gamma\right] \tag{5}$$

  assuming for all kinematically allowed values of t
  - t-independence of phase of $F^{\mathrm{N}}(s,t)$, i.e., quantity $\rho(t) = \frac{\mathrm{Re}\, F^{\mathrm{N}}(t)}{\mathrm{Im}\, F^{\mathrm{N}}(t)} = \mathrm{const}$
  - purely exponential $\left|F^{\mathrm{N}}(s,t)\right|$ in t, i.e., diffractive slope $B(t) = \frac{2}{|F^{\mathrm{N}}(t)|}\frac{\mathrm{d}}{\mathrm{d}t}\left|F^{\mathrm{N}}(t)\right| = \mathrm{const}$

- used widely in the era of ISR for determination of $\sigma^{\mathrm{tot,N}}$, quantity $\rho(t=0)$ and $B(t=0)$

Problems and limitations involved in the WY approach

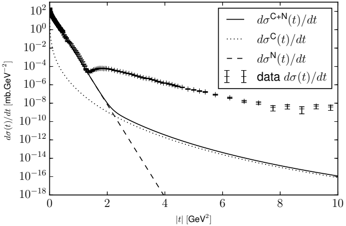
- relative phase $\phi(s,t)$ is real (defined as imaginary part of a complex function) $\Rightarrow$ the integral WY formula (3) consistent only with $\rho(t) = \text{const}$

  V. Kundrát, M. Lokajíček, and I. Vrkoč, "Limited validity of West and Yennie integral formula for elastic scattering of hadrons", Phys. Lett. **B656**, 182–185 (2007)

  J. Procházka and V. Kundrát, "Eikonal model analysis of elastic proton-proton collisions at 52.8 GeV and 8 TeV", arXiv:1606.09479 (2019)

- if $B(t)$ is t-independent $\Rightarrow$ contradiction to existence of observed dip-bump structure

  J. Kašpar, V. Kundrát, M. Lokajíček, and J. Procházka, "Phenomenological models of elastic nucleon scattering and predictions for LHC", Nucl.Phys. **B843**, 84–106 (2011)



- whole approach a priory limited and applied to data in region of only very small values of $|t|$ ($|t| \lesssim 0.01$ GeV^2 at 52.8 GeV)
- form factors $G_{1,2}(t)$ added by hand to the final interference formula(s)
- dependence of elastic hadronic collisions on impact parameter not considered
- ...

$\Rightarrow$ WY approach inapplicable for reliable data analysis; not usable for studying t-dependence of hadronic amplitude and b-dependent characteristics; see detailed discussion in, e.g., J. Procházka and V. Kundrát, "Eikonal model analysis of elastic proton-proton collisions at 52.8 GeV and 8 TeV", arXiv:1606.09479 (2019)

Additional comments to the WY approach

many descriptions of elastic scattering negativelly influenced by the simplified approach of WY

- quantity $\rho(t=0)$ and diffractive slope $B(t=0)$
  - unclear physical meaning (only very indirect relation to particle characteristics/interactions)
  - importance of these quantities overestimated in many contemporary hadronic models mainly under the influence of the WY approach where they are determining $F^{\mathrm{N}}(s,t)$ at all values of t - both quantities assumed, without any reasoning, to be t-independent at *all* kinematically allowed values of t, see page 5
- measured $\mathrm{d}\sigma/\mathrm{d}t$ commonly divided into two parts
  1. region of very low values of $|t|$ (e.g., $|t| \lesssim 0.01$ GeV^2 at 52.8 GeV)
     analyzed with the help of the simplified WY interference formula assuming specific t-dependence of $F^{\mathrm{N}}(s,t)$ at *all* values of t
  2. region of higher values of $|t|$ (containing dip-bump structure)
     described with the help of elastic hadronic models having different t-dependence of hadronic amplitude than the one assumed in the WY approach

  $\Rightarrow$ inconsistent dual description of data

$\Rightarrow$ one should look for different and more general description of (Coulomb-)hadronic elastic scattering; one should study transition from initial to final states, full physical picture

- t-dependence of $F^{\mathrm{N}}(s,t)$? i.e., t-dependences of hadronic modulus and phase?
  or, equivalently, t-dependences of quantities $\rho(t)$ and $B(t)$?
- b-dependent characteristics of collisions taking into account that initial states corresponding given value of b have different frequencies (weights)? (one should not mix characteristics of collisions at different impact parameter values)
- corresponding physical properties of colliding particles?

Eikonal model approach

- introduces dependence of elastic collisions on impact parameter
- several authors started from it or have been developing it (Glauber, van Hove, Miettinen, Islam, Cahn,...); results on various level of sophistication
- Coulomb-hadronic interference formula derived by Kundrát and Lokajíček (1994)

$$F^{C+N}_{eik}(s,t) = \pm\frac{\alpha s}{t} G_1(t) G_2(t) + F^{N}(s,t)[1 \mp i\alpha \bar{G}(s,t)] \tag{6}$$

where

$$\bar{G}(s,t) = \int_{t_{min}}^{0} dt' \left\{ \ln\left(\frac{t'}{t}\right) \frac{d}{dt'}[G_1(t')G_2(t')] - \frac{1}{2\pi}\left[\frac{F^N(s,t')}{F^N(s,t)} - 1\right] I(t,t') \right\} \tag{7}$$

and

$$I(t,t') = \int_0^{2\pi} d\Phi'' \frac{G_1(t'')G_2(t'')}{t''}. \tag{8}$$

- derived for *any* t and s (high energy) with the aim *not* to impose any restriction on t-dependence of $F^N(s,t)$
- allows description of data in the *whole* measured t-range (NB: to calculate $F^{C+N}(s,t)$ at given value of t needs to be known $F^N(s,t)$ at *all* values of t - even *outside* measured t-range) $\Rightarrow$ consistent description of data (no duality)

Definition: central vs. peripheral behaviour of elastic collisions

- Definition: two basic types of behaviour of elastic hadron collisions (models) in dependence on impact parameter may be distinguished

  1. peripheral: $\sqrt{\langle b^2 \rangle^{\text{el}}} > \sqrt{\langle b^2 \rangle^{\text{inel}}}$
     i.e., if elastic collisions correspond in average to higher impact parameter b then the inelastic ones; corresponds to usual ideas of collisions of two matter objects
  2. central: $\sqrt{\langle b^2 \rangle^{\text{el}}} < \sqrt{\langle b^2 \rangle^{\text{inel}}}$
     the opposite; anti-ontological behaviour; some kind of transparency of colliding particles; corresponding particle structure never sufficiently explained in the literature

Elastic hadronic amplitude $F^{\mathrm{N}}(s,t)$

1. elastic hadronic amplitude in many contemporary models is a priory strongly constrained from the very beggining without sufficient reasoning by requiring:
   1.1 dominance of the imaginary part of $F^{\mathrm{N}}(s,t)$ in quite broad interval of t in forward region
   1.2 vanishing of the imaginary part of $F^{\mathrm{N}}(s,t)$ at (or around) $t = t_{\mathrm{dip}}$ (wrongly reasoned as a consequence of the minimum of $\mathrm{d}\sigma/\mathrm{d}t$ at t_{dip})
   1.3 change of sign of the real part of $F^{\mathrm{N}}(s,t)$ at "low" value of $|t|$ (required by Martin's theorem [8] derived under certain conditions)
   1.4 values of $\sigma^{\mathrm{tot,N}}$, $B(t=0)$ and $\rho(t=0)$ obtained from the simplified formula of WY (misleadingly denoted as "measurement", see page (33))

   the corresponding t-dependence of $F^{\mathrm{N}}(s,t)$ (its phase) is strongly constrained by these requirements and it may be shown that mainly the first requirement leads to central behaviour of elastic collisions, see page (29)
2. one may ask if it is possible to obtain description of data which would lead to peripheral behaviour of elastic collisions (without imposing the unreasoned constrains above);
   1981 [9] - peripheral solution of the scattering problem may be obtained if hadronic phase has specific t-dependence

$\Rightarrow$ one may try to determine $F^{\mathrm{N}}(s,t)$ on the basis of experimental data under given set of assumptions (constrains) and study their impact on values of determined hadronic quantities

Analysis of measured elastic pp data I

- eikonal Coulomb-hadronic interference formula used for determination of hadronic amplitude from measured data

$$\frac{\mathrm{d}\sigma(s,t)}{\mathrm{d}t} = \frac{\pi}{sp^2}\left|F^{C+N}_{eik}(s,t)\right|^2 \tag{9}$$

- hadronic amplitude may be parameterized

$$F^{\mathrm{N}}(s,t) = \mathrm{i}\left|F^{\mathrm{N}}(s,t)\right| \mathrm{e}^{-\mathrm{i}\zeta^{\mathrm{N}}(s,t)} \tag{10}$$

- modulus (very general parameterization)

$$\left|F^{\mathrm{N}}(s,t)\right| = (a_1 + a_2 t)\,\mathrm{e}^{b_1 t + b_2 t^2 + b_3 t^3} + (c_1 + c_2 t)\,\mathrm{e}^{d_1 t + d_2 t^2 + d_3 t^3} \tag{11}$$

- very general phase parameterization (analytic if κ is positive integer)

$$\zeta^{\mathrm{N}}(s,t) = \zeta_0 + \zeta_1 \left(\frac{t}{t_0}\right)^{\kappa} \mathrm{e}^{\nu t} \tag{12}$$

it may reproduce various t shapes in dependence on values of the free parameters (according to additional constrains)

Analysis of measured elastic pp data II

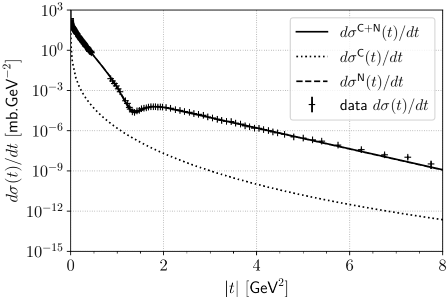
several fits of measured $d\sigma/dt$ at 52.8 GeV and 8 TeV performed under different assumptions:

1. different choices of form factors (effective electric vs. effective electromagnetic)
2. different constrains of hadronic amplitude $F^{N}(s,t)$
   - 2.1 widely used strong constrains imposed on $F^{N}(s,t)$
     - possible t-dependences of hadronic phase a priory strongly limited
       $\Rightarrow$ unique solution of the t-dependence of $F^{N}(s,t)$ determined from data
     - mainly the required dominance of the imaginary part of $F^{N}(s,t)$ leads to centrality of elastic collisions;
   - 2.2 alternative peripheral case
     - used parameterization of the phase (12) allowing very different t-dependences and required peripherality
     - no unique solution of the t-dependence of $F^{N}(s,t)$ determined from data
       $\Rightarrow$ added additional constrain on required value of $\sqrt{\langle b^2 \rangle^{\rm el}}$ and 3 different alternatives shown
     - needed to solve complicated problem of bounded extrema (non-trivial optimization problem)

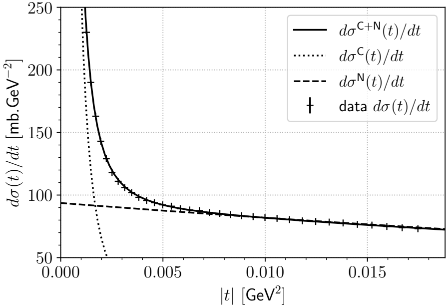
4 different fits/models of data at each energy showed in the following (1 central, 3 different peripheral alternatives)

- effective *electromagnetic* form factors - Fit 1b central, Fits. 2b-4b peripheral

Results at 52.8 GeV - fitted $\mathrm{d}\sigma/\mathrm{d}t$



(a) full available $|t|$-range of measured data

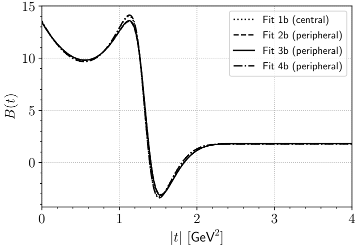


(b) region of very low values of $|t|$

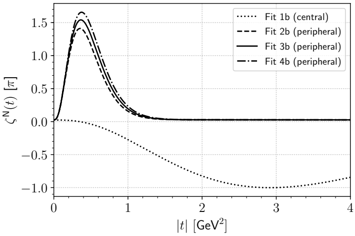
- fits in the very broad interval $|t| \in \langle 0.00126, 7.75 \rangle$ GeV^2 including *both* peak at the lowest measured values of $|t|$ and dip-bump structure at higher values of $|t|$
- all the performed fits at 52.8 GeV lead to similar t-dependences of hadronic $\mathrm{d}\sigma/\mathrm{d}t$

$\sqrt{s}$	[GeV]	52.8	52.8	52.8	52.8
Fit		1b	2b	3b	4b
Case		central	peripheral	peripheral	peripheral
Form factor		effective electromagnetic	effective electromagnetic	effective electromagnetic	effective electromagnetic
ζ_0		0.0762 ± 0.0017	0.0824 ± 0.0017	0.0825 ± 0.0017	0.0849 ± 0.0017
ζ_1		-2.605	1970 ± 37	1974 ± 37	1964 ± 38
κ		3	3	3	3
ν	[GeV^{-2}]	1.028	8.48 ± 0.13	8.23 ± 0.14	8.02 ± 0.15
a_1		12149.8 ± 9.2	12203 ± 10	12202.3 ± 9.3	12225 ± 30
a_2	[GeV^{-2}]	10705 ± 29	10760 ± 39	10767 ± 33	10760 ± 150
b_1	[GeV^{-2}]	5.905 ± 0.017	5.867 ± 0.018	5.868 ± 0.017	5.934 ± 0.021
b_2	[GeV^{-4}]	3.677 ± 0.063	3.440 ± 0.062	3.445 ± 0.060	3.722 ± 0.080
b_3	[GeV^{-6}]	1.678 ± 0.041	1.518 ± 0.040	1.520 ± 0.038	1.698 ± 0.062
c_1		58.8 ± 1.4	60.6 ± 3.0	60.4 ± 1.9	57.4 ± 26
c_2	[GeV^{-2}]	-5.4e-6 ± 2.9	-1.8e-5 ± 2.4	-6.3e-8 ± 2.3	-1.4 ± 4.4
d_1	[GeV^{-2}]	0.901 ± 0.050	0.908 ± 0.042	0.907 ± 0.041	0.918 ± 0.039
$\rho(t=0)$		0.0763 ± 0.0017	0.0826 ± 0.0017	0.0827 ± 0.0016	0.0851 ± 0.017
$B(t=0)$	[GeV^{-2}]	13.515 ± 0.035	13.439 ± 0.037	13.444 ± 0.036	13.573 ± 0.055
$\sigma^{\mathrm{tot,N}}$	[mb]	42.694 ± 0.033	42.864 ± 0.037	42.861 ± 0.034	42.917 ± 0.14
$\sigma^{\mathrm{el,N}}$	[mb]	7.469	7.542	7.539	7.532
σ^{inel}	[mb]	35.22	35.32	35.32	35.39
$\sigma^{\mathrm{el,N}}/\sigma^{\mathrm{tot,N}}$		0.1750	0.1759	0.1759	0.1755
$\mathrm{d}\sigma^{\mathrm{N}}/\mathrm{d}t(t=0)$	[mb.GeV^{-2}]	93.67	94.52	94.51	94.80
$\sqrt{\langle b^2\rangle^{\mathrm{tot}}}$	[fm]	1.026	1.023	1.023	1.028
$\sqrt{\langle b^2\rangle^{\mathrm{el}}}$	[fm]	0.6778	1.854	1.959	2.045
$\sqrt{\langle b^2\rangle^{\mathrm{inel}}}$	[fm]	1.085	0.7322	0.671	0.6261
$D^{\mathrm{tot}}(b=0)$		1.29	1.30	1.30	1.29
$D^{\mathrm{el}}(b=0)$		0.530	0.0317	0.0342	0.0466
$D^{\mathrm{inel}}(b=0)$		0.762	1.27	1.27	1.25

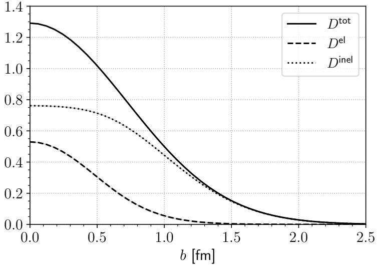
Results at 52.8 GeV - t and b dependent characteristics



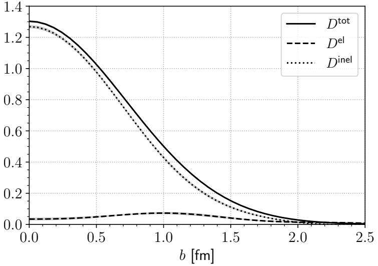
(a) diffractive slopes



(b) hadronic phases

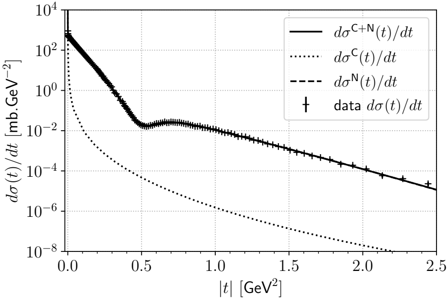


(c) profile functions: central case, Fit 1b

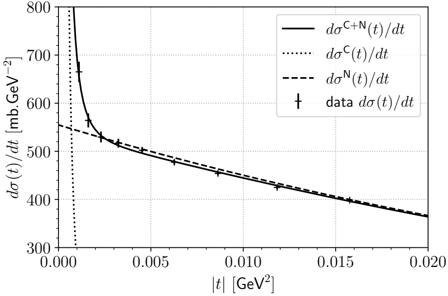


(d) profile functions: peripheral case, Fit 3b

Results at 8 TeV - fitted $d\sigma/dt$



(a) full available $|t|$-range of measured data

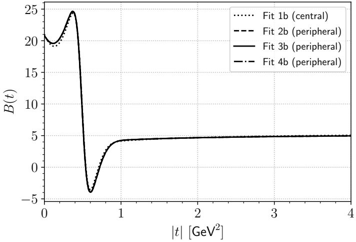


(b) region of very low values of $|t|$

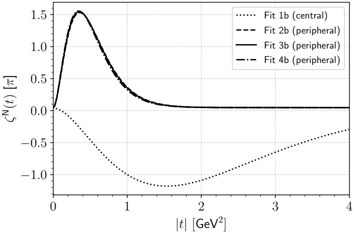
- TOTEM 8 TeV data (1000m and 90m optics data up to $|t| = 0.2$ GeV^2, see [10, 11]); extended by renormalized TOTEM 7 TeV data [12] up to 2.5 GeV^2 to obtain data in wider t-region ($|t| \in \langle 6 \times 10^{-4}, 2.5 \rangle$ GeV^2) including both region of peak at very low values of $|t|$ and dip-bump structure region $\Rightarrow$ approximate data denoted as "8 TeV data"
- all the performed fits at 8 TeV leads to similar t-dependences of hadronic $d\sigma/dt$

$\sqrt{s}$	[GeV]	8000	8000	8000	8000
Fit		1b	2b	3b	4b
Case		central	peripheral	peripheral	peripheral
Form factor		effective electromagnetic	effective electromagnetic	effective electromagnetic	effective electromagnetic
ζ_0		0.121 ± 0.018	0.147 ± 0.017	0.148 ± 0.016	0.147 ± 0.017
ζ_1		-12.02	270 ± 11	281 ± 11	298 ± 11
κ		2	2	2	2
ν	[GeV^{-2}]	1.304	5.60 ± 0.20	5.68 ± 0.20	5.83 ± 0.19
a_1	[10^{-7}]	66.58 ± 0.12	66.74 ± 0.15	66.79 ± 0.11	66.87 ± 0.21
a_2	[10^{-7}GeV^{-2}]	163.06 ± 0.73	170.35 ± 0.49	170.39 ± 0.39	170.44 ± 0.53
b_1	[GeV^{-2}]	8.291 ± 0.038	8.13 ± 0.034	8.137 ± 0.026	8.153 ± 0.029
b_2	[GeV^{-4}]	9.27 ± 0.23	7.57 ± 0.20	7.58 ± 0.16	7.62 ± 0.17
b_3	[GeV^{-6}]	14.85 ± 0.34	12.13 ± 0.30	12.15 ± 0.25	12.20 ± 0.27
c_1	[10^{-7}]	1.57 ± 0.14	2.06 ± 0.12	2.047 ± 0.067	2.03 ± 0.17
c_2	[10^{-7}GeV^{-2}]	-3.14 ± 0.33	-2.43 ± 0.32	-2.46 ± 0.14	-2.51 ± 0.44
d_1	[GeV^{-2}]	2.75 ± 0.077	2.683 ± 0.045	2.688 ± 0.019	2.700 ± 0.060
$\rho(t=0)$		0.122 ± 0.018	0.148 ± 0.019	0.149 ± 0.016	0.148 ± 0.017
$B(t=0)$	[GeV^{-2}]	21.021 ± 0.085	20.811 ± 0.017	20.829 ± 0.055	20.859 ± 0.073
$\sigma^{\mathrm{tot,N}}$	[mb]	103.44 ± 0.35	104.07 ± 0.38	104.12 ± 0.31	104.2 ± 0.48
$\sigma^{\mathrm{el,N}}$	[mb]	27.6	28.0	28.0	28.0
σ^{inel}	[mb]	75.9	76.1	76.1	76.2
$\sigma^{\mathrm{el,N}}/\sigma^{\mathrm{tot,N}}$		0.267	0.269	0.269	0.269
$\mathrm{d}\sigma^{\mathrm{N}}/\mathrm{d}t(t=0)$	[mb.GeV^{-2}]	555	566	566	567
$\sqrt{\langle b^2\rangle^{\mathrm{tot}}}$	[fm]	1.28	1.27	1.27	1.27
$\sqrt{\langle b^2\rangle^{\mathrm{el}}}$	[fm]	0.896	1.83	1.86	1.91
$\sqrt{\langle b^2\rangle^{\mathrm{inel}}}$	[fm]	1.39	0.992	0.970	0.937
$D^{\mathrm{tot}}(b=0)$		2.01	2.04	2.04	2.04
$D^{\mathrm{el}}(b=0)$		0.980	0.216	0.205	0.190
$D^{\mathrm{inel}}(b=0)$		1.03	1.83	1.84	1.85

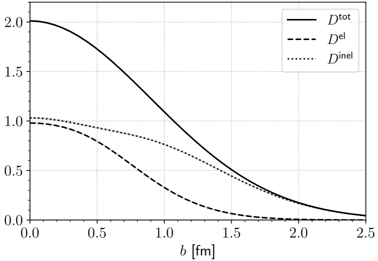
Results at 8 TeV - t and b dependent characteristics



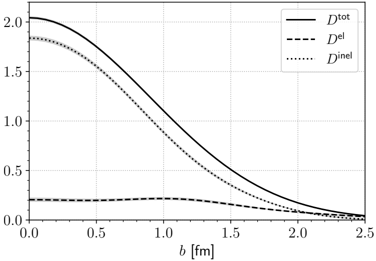
(a) diffractive slopes



(b) hadronic phases



(c) profile functions: central case, Fit 1b



(d) profile functions: peripheral case, Fit 3b

Summary and conclusion I

1. WY approach
   - used widely at ISR for "measurement" of $\sigma^{\text{tot,N}}$, $B(t=0)$ and $\rho(t=0)$
   - many problems and limitations identified later (several papers exist)
   - ⇒ WY approach should be abandoned in the era of LHC as it may lead to wrong physical conclusions; it should not be used for constraining hadronic models based on assumptions inconsistent with the simplified model of WY
   - ⇒ one should look for other description of el. scattering of (charged) hadrons
2. eikonal model approach
   - more general and relevant for analysis of el. data than the (over)simplified approach of WY
   - more fundamental then other contemporary models of el. scattering as it may be used for description of Coulomb-hadronic interference *and* take into account also dependence of collisions on impact parameter (with the aim not to mix collisions corresponding to different values of impact parameter)
   - elastic scattering data at 52.8 GeV and 8 TeV analyzed under different assumptions consistently in the whole measured t-range and corresponding results compared
   - the results obtained with the help of the eikonal model represent the most detailed and elaborated impact parameter analysis of elastic pp collision data which has ever been performed
   - ⇒ transparency of protons during elastic collisions based on unreasoned assumptions; corresponding structure of protons never sufficiently explained in the literature
   - ⇒ elastic collision process may be interpreted as peripheral and protons as compact (non-transparent) particles

- J. Procházka and V. Kundrát, "Eikonal model analysis of elastic proton-proton collisions at 52.8 GeV and 8 TeV", arXiv:1606.09479 (2019)
- J. Procházka, "Elastic proton-proton collisions at high energies", CERN-THESIS-2018-294, PhD thesis (Feb. 2018)
- TOTEM Collaboration, "Measurement of elastic pp scattering at $\sqrt{s}=8$ TeV in the Coulomb-nuclear interference region – determination of the ρ-parameter and the total cross-section", Eur. Phys. J. C **76**, 661 (2016), see also CERN-PH-EP-2015-325, arXiv:1610.00603

Summary and conclusion II

Identified deeper problems and open questions in <u>all</u> contemporary descriptions of el. scattering:

1. Coulomb interaction and experimental conditions
   1.1 (non-)divergence at $t = 0$
   1.2 multiple collisions
   1.3 electromagnetic form factors
2. different mechanism of Coulomb and strong forces
3. weak (contact) interaction
4. properties of S matrix and structure of Hilbert space
5. optical theorem
6. determination of b-dependent probability functions of hadron collisions
7. distribution of elastic scattering angles for a given value of impact parameter
8. increase of integrated total, elastic and inelastic cross sections and dimensions of colliding particles in dependence on collision energy
9. extrapolations outside measured regions

- main results concerning b-dependence of elastic collisions and the identified open problems (including historical context) published

  J. Procházka, M. V. Lokajíček, and V. Kundrát, "Dependence of elastic hadron collisions on impact parameter", Eur. Phys. J. Plus **131**, 147 (2016), see also arXiv:1509.05343 (2015)
- problems related specifically to optical theorem discussed in

  J. Procházka, V. Kundrát, and M. V. Lokajíček, "Elastic scattering of hadrons without optical theorem", arXiv: 1502.00468 (2015)
- points 8. and 9. discussed in

  J. Procházka and V. Kundrát, "Eikonal model analysis of elastic proton-proton collisions at 52.8 GeV and 8 TeV", arXiv:1606.09479 (2019)

Summary and conclusion III

- Proper analysis of hadron collisions in dependence on impact parameter may provide important insight concerning shapes and dimensions (and other properties) of collided particles which can be hardly obtained in a different way. One should carefully study the assumptions involved in any collision model and test the consequences. It is also necessary to solve already known fundamental problems and open questions in *all* contemporary descriptions of elastic pp scattering (this includes Regge approach, eikonal model, WY, ... - see page (20)) before making far-reaching conclusions concerning structure and properties of collided particles.
- further comments and new ideas how to move forward

1. J. Procházka, M. V. Lokajíček, and V. Kundrát, "Dependence of elastic hadron collisions on impact parameter", Eur. Phys. J. Plus **131**, 147 (2016), see also arXiv:1509.05343 (2015)
2. M. V. Lokajíček, V. Kundrát, and J. Procházka, "Schrödinger equation and (future) quantum physics", in *Advances in quantum mechanics*, edited by P. Bracken (InTech Publisher, 2013), pp. 105–132

- even more fundamental and broader comments related to contemporary state of fundamental physical research + historical context

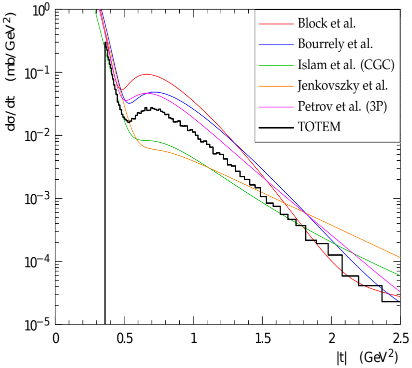
M. V. Lokajíček and J. Procházka, "The contemporary state of fundamental physical research and the future path to scientific knowledge", arXiv:1610.08331 (2019)

$\Rightarrow$ to make progress in physics one needs to return to

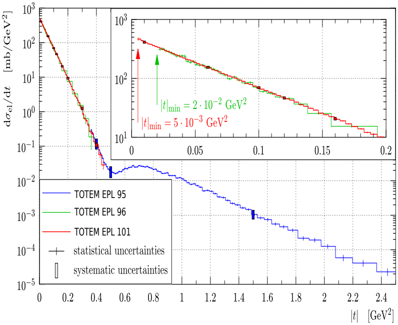
- causal ontology
- falsification approach
  - logic, require logical consistence of any description of physical reality
  - systematical analysis of involved assumptions, testing,...

Backup

Measured elastic $d\sigma/dt$ at $\sqrt{s} = 7$ TeV by TOTEM



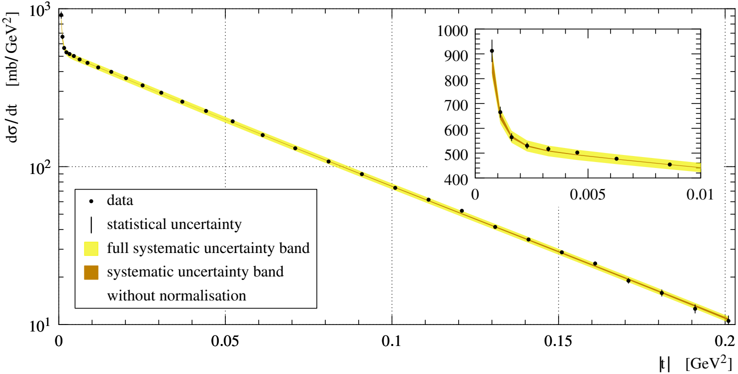
(a) first measurement of pp elastic differential cross section by TOTEM and predictions of different phenomenological models [18]



(b) three (independent) measurements by TOTEM at different settings [12]

- elastic pp differential cross section at energy of $\sqrt{s} = 7$ TeV has similar t-dependence as at the ISR energies
- dip-bump structure in pp data observed again by TOTEM at the LHC since the era of the ISR

Measured elastic $d\sigma/dt$ at $\sqrt{s} = 8$ TeV by TOTEM



- significantly non-exponential part (see the points corresponding to the lowest measured values of $|t|$) attributed standardly to Coulomb-hadronic interference; special beam optics ($\beta^* = 1000$ m) has been needed to reach such low values of $|t|$ [10]

Comparison of proton electromagnetic form factors

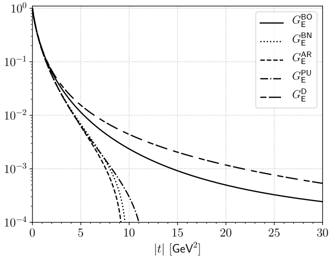


Figure: electric form factors

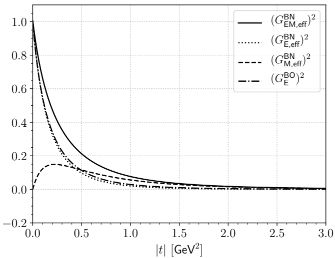


Figure: effective form factors

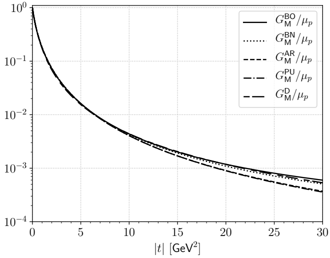


Figure: magnetic form factors

- different authors/analyses
  - different shapes of form factors
- electric form factors
  - differences visible at $|t| > 2.5$ GeV^2
- electric vs. effective electromagnetic form factors
  - very significant differences already at lower $|t|$ values
  - originally only the electric form factors used in the eikonal interference formula
  - impact on determination of $F^N(s,t)$ if effective electromagnetic form factors are used instead of electric form factors?

Hadronic quantities I

- modulus and phase of hadronic amplitude may be defined as

$$F^{\mathrm{N}}(s,t) = \mathrm{i} \left| F^{\mathrm{N}}(s,t) \right| \mathrm{e}^{-\mathrm{i}\zeta^{\mathrm{N}}(s,t)} \tag{13}$$

  it means

$$\tan \zeta^{\mathrm{N}}(s,t) = \rho(s,t) \tag{14}$$

- t-dependent quantities may be introduced

$$B(s,t) = \frac{\mathrm{d}}{\mathrm{d}t} \left[\ln \frac{\mathrm{d}\sigma^{\mathrm{N}}}{\mathrm{d}t}(s,t) \right] = \frac{2}{\left| F^{\mathrm{N}}(s,t) \right|} \frac{\mathrm{d}}{\mathrm{d}t} \left| F^{\mathrm{N}}(s,t) \right| \tag{15}$$

$$\rho(s,t) = \frac{\mathrm{Re}\, F^{\mathrm{N}}(s,t)}{\mathrm{Im}\, F^{\mathrm{N}}(s,t)} \tag{16}$$

Several physically interesting quantities derived from the hadronic amplitude $F^{\mathrm{N}}(s,t)$

- total cross section (optical theorem)

$$\sigma^{\mathrm{tot,N}}(s) = \frac{4\pi}{p\sqrt{s}} \, \mathrm{Im}\, F^{\mathrm{N}}(s, t=0) \tag{17}$$

- elastic and inelastic cross sections

$$\sigma^{\mathrm{el,N}} = \int \frac{\mathrm{d}\sigma^{\mathrm{el,N}}}{\mathrm{d}t} = \int \frac{\pi}{sp^2} \left| F^{\mathrm{N}}(s,t) \right|^2 ; \qquad \sigma^{\mathrm{inel}} = \sigma^{\mathrm{tot,N}} - \sigma^{\mathrm{el,N}} \tag{18}$$

Hadronic quantities II

- elastic hadronic amplitude in b-space - Fourier-Bessel (FB) transformation (Adachi, Kotani, Takeda, Islam, ...)

$$h_{\mathrm{el}}(s,b) = h_1(s,b) + h_2(s,b)$$
$$= \frac{1}{4p\sqrt{s}} \int_{t_{\min}}^{0} F^{\mathrm{N}}(s,t) J_0(b\sqrt{-t}) \mathrm{d}t + \frac{1}{4p\sqrt{s}} \int_{-\infty}^{t_{\min}} \lambda(s,t) J_0(b\sqrt{-t}) \mathrm{d}t \tag{19}$$

- unitarity condition at *finite* energies

$$\mathrm{Im}\, h_1(s,b) + c(s,b) = |h_1(s,b)|^2 + g_1(s,b) + K(s,b) + c(s,b) \tag{20}$$

- profile functions
  - main b-dependent characteristics of collisions, introduced in analogy to description of some optics phenomena (light meeting an obstacle of a given profile which describes its absorptive properties)
  - sometimes interpreted as probabilities of total, elastic or inelastic collision at given value of impact parameter
  -

$$D^{\mathrm{el}}(s,b) \equiv 4\,|h_1(s,b)|^2, \tag{21}$$
$$D^{\mathrm{tot}}(s,b) \equiv 4\,(\mathrm{Im}\, h_1(s,b) + c(s,b)), \tag{22}$$
$$D^{\mathrm{inel}}(s,b) \equiv 4\,(g_1(s,b) + K(s,b) + c(s,b)) \tag{23}$$

- cross sections determined on the basis of the profile functions (X=tot, el, inel)

$$\sigma^{\mathrm{X}}(s) = 2\pi \int_0^{\infty} b\mathrm{d}b\, D^{\mathrm{X}}(s,b). \tag{24}$$

Hadronic quantities III

mean-square values of impact parameter b

- definition ($n = 2$ and $w(b) = 2\pi b$)

$$\langle b^n \rangle^{\mathrm{X}} = \frac{\int\limits_0^\infty b^n \, w(b) D^{\mathrm{X}}(s,b) \mathrm{d}b}{\int\limits_0^\infty w(b) D^{\mathrm{X}}(s,b) \mathrm{d}b} \tag{25}$$

- expressions of the mean-square values in terms of $F^{\mathrm{N}}(s,t)$

  V. Kundrát, M. V. Lokajíček, and D. Krupa, "Impact parameter structure derived from elastic collisions", Phys. Lett. B **544**, 132–138 (2002)

$$\begin{aligned}
\langle b^2 \rangle^{\mathrm{el}} =& \langle b^2 \rangle^{\mathrm{mod}} + \langle b^2 \rangle^{\mathrm{ph}} \\
=& \frac{4 \int\limits_{t_{\min}}^{0} \mathrm{d}t |t| \left(\frac{\mathrm{d}}{\mathrm{d}t} \left| F^{\mathrm{N}}(s,t) \right| \right)^2}{\int\limits_{t_{\min}}^{0} \mathrm{d}t \left| F^{\mathrm{N}}(s,t) \right|^2} + \frac{4 \int\limits_{t_{\min}}^{0} \mathrm{d}t \left| F^{\mathrm{N}}(s,t) \right|^2 |t| \left(\frac{\mathrm{d}}{\mathrm{d}t} \zeta^{\mathrm{N}}(s,t) \right)^2}{\int\limits_{t_{\min}}^{0} \mathrm{d}t \left| F^{\mathrm{N}}(s,t) \right|^2} \\
\langle b^2 \rangle^{\mathrm{tot}} =& 4 \left(\frac{\frac{\mathrm{d}}{\mathrm{d}t} \left| F^{\mathrm{N}}(s,t) \right|}{\left| F^{\mathrm{N}}(s,t) \right|} - \tan \zeta^{\mathrm{N}}(s,t) \frac{\mathrm{d}}{\mathrm{d}t} \zeta^{\mathrm{N}}(s,t) \right) \Bigg|_{t=0} \\
\langle b^2 \rangle^{\mathrm{inel}} =& \frac{\sigma^{\mathrm{tot,N}}(s) \langle b^2 \rangle^{\mathrm{tot}} - \sigma^{\mathrm{el,N}}(s) \langle b^2 \rangle^{\mathrm{el}}}{\sigma^{\mathrm{inel}}(s)}
\end{aligned} \tag{26}$$

Contemporary phenomenological models of ES

- J. Kašpar, V. Kundrát, M. Lokajíček, and J. Procházka, "Phenomenological models of elastic nucleon scattering and predictions for LHC", Nucl.Phys. **B843**, 84–106 (2011)

Model	theoretical framework	$\sqrt{\langle b^2 \rangle^{tot}}$	$\sqrt{\langle b^2 \rangle^{el}}$	$\sqrt{\langle b^2 \rangle^{inel}}$
Bourelly et al.	eikonal	1.249	0.876	1.399
Petrov et al. (2P)	eikonal+Regge	1.227	0.875	1.324
Petrov et al. (3P)	eikonal+Regge	1.263	0.901	1.375
Block et al.	eikonal (QCD-inspired)	1.223	0.883	1.336
Islam et al.	eikonal	1.552	1.048	1.659

Table: Values of root-mean-squares of impact parameter (in femtometers) predicted by several contemporary phenomenological models of pp collisions at collision energy of 14 TeV [7, 20].

all the models predict $\sqrt{\langle b^2 \rangle^{el}} < \sqrt{\langle b^2 \rangle^{inel}}$, i.e., *central* behaviour of elastic hadronic scattering

- it is mainly the dominance of imaginary part of $F^N(s,t)$ (assumed, without any justification, in many contemporary models of elastic scattering) which leads to the centrality

Real and imaginary parts of $F^{N}(s,t)$ at 52.8 GeV

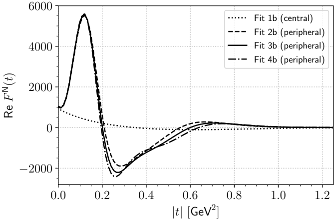


Figure: real parts - Fits. 1b-4b

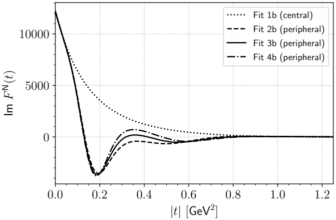


Figure: imaginary parts - Fits. 1b-4b

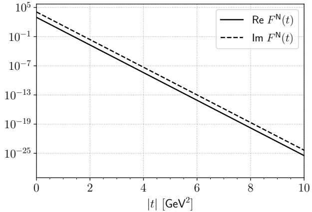


Figure: WY - real and imaginary parts (corresponding free parameters of $F^{N}(s,t)$ taken from [21])

- Eikonal model
  - Fit 1b (central) - real part of $F^{N}(s,t)$ changes sign at $|t| \approx 0.35$ GeV^2
  - Fits. 2b-4b (peripheral) - real part of $F^{N}(s,t)$ changes sign at $|t| \approx 0.2$ GeV^2

  $\Rightarrow$ conclusion of the Martin's theorem *fulfilled*
- WY
  - real part of $F^{N}(s,t)$ does not change sign at any t value (hadronic phase assumed to be t-independent)

  $\Rightarrow$ conclusion of the Martin's theorem *not fulfilled*

Real and imaginary parts of $F^{\mathrm{N}}(s,t)$ at 8 TeV

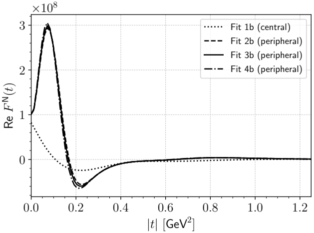


Figure: real parts - Fits. 1b-4b

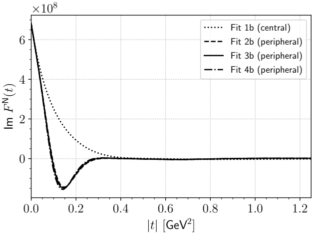


Figure: imaginary parts - Fits. 1b-4b

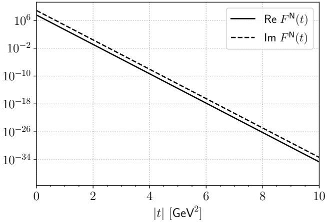


Figure: WY - real and imaginary parts (corresponding free parameters of $F^{\mathrm{N}}(s,t)$ taken from [10])

- Eikonal model
  - Fit 1b (central) - real part of $F^{\mathrm{N}}(s,t)$ changes sign at $|t| \approx 0.1$ GeV^2
  - Fits. 2b-4b (peripheral) - real part of $F^{\mathrm{N}}(s,t)$ changes sign at $|t| \approx 0.18$ GeV^2

  $\Rightarrow$ conclusion of the Martin's theorem *fulfilled*
- WY
  - real part of $F^{\mathrm{N}}(s,t)$ does not change sign at any $|t|$ value (hadronic phase assumed to be t-independent)

  $\Rightarrow$ conclusion of the Martin's theorem *not fulfilled*

Eikonal model - summary of the obtained results

1. measured $d\sigma/dt$ at two very different energies 52.8 GeV and 8 TeV analyzed in broad interval of $|t|$ values (including both peak at low values of $|t|$ and also region of dip-bump structure at higher $|t|$ values) under different assumptions (constrains) to determine their impact on hadronic quantities (and overall relevance of the given description)
2. for each description the corresponding t-dependence of $F^{N}(s,t)$ at *all* values of t has been determined from data and several quantities characterizing the collision process in t and b space have been calculated - to study the whole physical picture at given energy under the given set of assumptions
3. the results show:
   - choice of form factor (effective electric vs. effective electromagnetic) - *small or negligible* impact on determination of hadronic amplitude
   - choice of t-dependence of hadronic phase - may *completely* change behaviour of collisions in b-space; the phase is only weakly constrained by the eikonal interference formula
   - t-dependence of $\left|F^{N}(s,t)\right|$ - *strongly* constrained by measured $d\sigma/dt$
4. hadronic amplitude in many models of elastic hadronic collisions is strongly a priory constrained without sufficient reasoning - our results show that these models then leads to *central* behaviour of elastic collisions (mainly due to required dominance of the imaginary part of $F^{N}(s,t)$ in forward region), corresponding particle structure has never been sufficiently explained in the literature
5. elastic collisions may be interpreted as *peripheral* processes (in agreement with usual ideas corresponding to collisions of two matter objects) just by allowing hadronic phase to be strongly t-dependent already at low values of $|t|$
6. corresponding hadronic amplitudes in all the studied alternatives are analytic, satisfy condition of unitarity and conclusion of Martin's theorem

Fitting of observed $\mathrm{d}\sigma/\mathrm{d}t(s,t)$ and "measured" $\sigma^{\mathrm{tot,N}}(s)$, $B(s,t\!=\!0)$, $\rho(s,t\!=\!0)$,...

- $\mathrm{d}\sigma/\mathrm{d}t(s,t)$ represents experimental data in the case of elastic scattering (ES)
- hadronic quantities like $\sigma^{\mathrm{tot,N}}$, $B(t\!=\!0)$, $\rho(t\!=\!0)$, ... are determined with the help of a model of ES applied to measured $\mathrm{d}\sigma/\mathrm{d}t(s,t)$ under several strong assumptions $\Rightarrow$ speaking about "measurement" of these quantities may be very misleading
- a model of ES ($F^{\mathrm{N}}(s,t)$) which is trying to describe given set of "measured" values should be consistent with the method/model/assumptions used to produce them (this is often not the case)
- it is typically not a priory clear how much given principle, theorem or other assumption constrain $F^{\mathrm{N}}(s,t)$, it may or may not be possible to describe experimental data under given set of assumptions - it needs to be study
- it is not sufficient to fit data $\mathrm{d}\sigma/\mathrm{d}t$ only at low values of $|t|$; one should study full physical picture (see also any Coulomb-hadronic formula and the FB transformation connecting t and b dependent amplitudes)

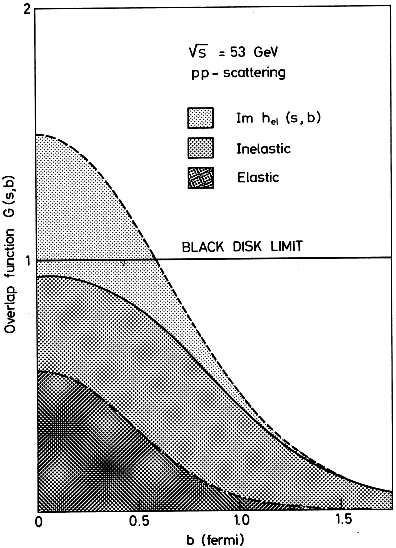
studies of method of measurement of elastic pp scattering ($\mathrm{d}\sigma/\mathrm{d}t(s,t)$) and various models of ES together with their assumptions are essential for determination of several hadronic characteristics which can be hardly obtained in a different way

Miettinen's results - profile functions at 52.8 GeV

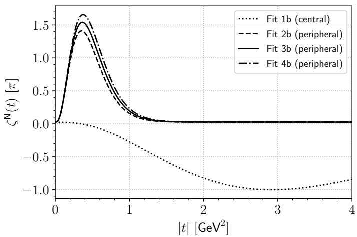
Miettinen [22–24] (1973-1975) - **one of the first discussions of behaviour of pp collisions in *b*-space** (many contemporary models are based on very similar calculations)

- profile functions (overlap functions $G(s, b)$ in Miettinen's terminology) established from pp data at 52.8 GeV and identified with probability functions
- it has been concluded from inelastic overlap (probability) function that $P^{\text{el}}(b=0) = 1 - P^{\text{inel}}(b=0) = 6\%$ (subtraction from "black disc limit")
- why the elastic probability $P^{\text{el}}(b=0)$ is not taken directly from the elastic overlap (probability) function which is $\approx 50\%$, i.e., completely different?
- why $\text{Im}\, h_{\text{el}}(s, b)$, which is often identified with total profile function, i.e., probability in this case, is significantly greater then 1 for certain values of b?

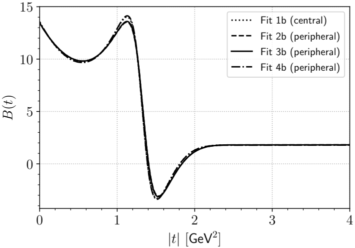
$\Rightarrow$ the results suggesting transparency of colliding particles during collisions should be strongly questioned or even doubted, see detailed discussion in J. Procházka, M. V. Lokajíček, and V. Kundrát, "Dependence of elastic hadron collisions on impact parameter", Eur. Phys. J. Plus **131**, 147 (2016), see also arXiv:1509.05343 (2015)



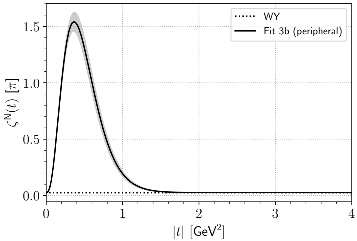
Results at 52.8 GeV - hadronic phase and diffractive slope



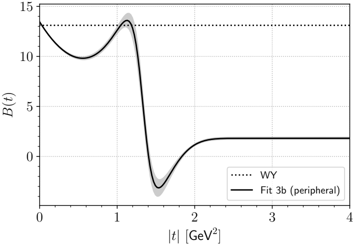
(a) hadronic phases



(b) diffractive slopes

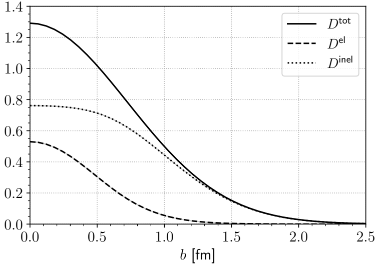


(c) hadronic phases - Fit 3b vs. WY

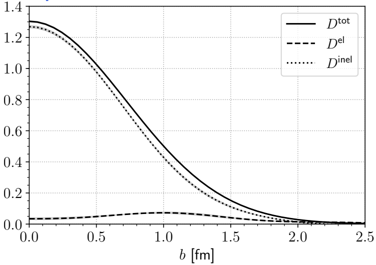


(d) diffractive slopes - Fit 3b vs. WY

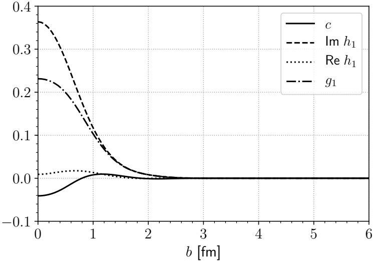
Results at 52.8 GeV - *b*-dependent functions



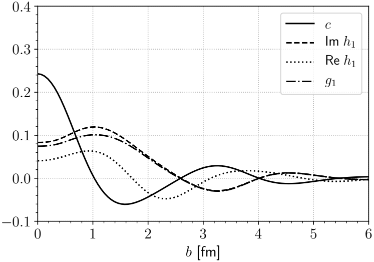
(a) profile functions: central case, Fit 1b



(b) profile functions: peripheral case, Fit 3b

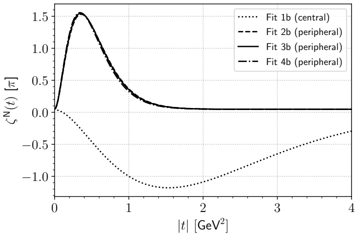


(c) central case - Fit 1b

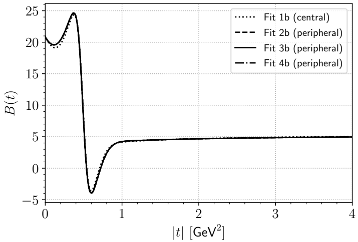


(d) peripheral case - Fit 3b

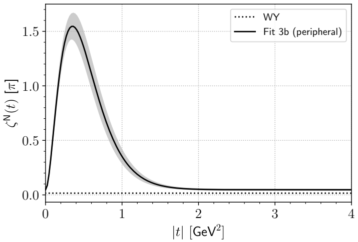
Results at 8 TeV - hadronic phase and diffractive slope



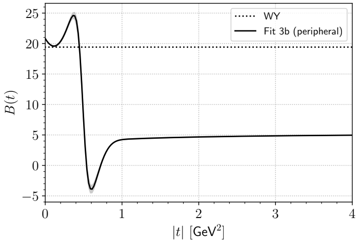
(a) hadronic phases



(b) diffractive slopes

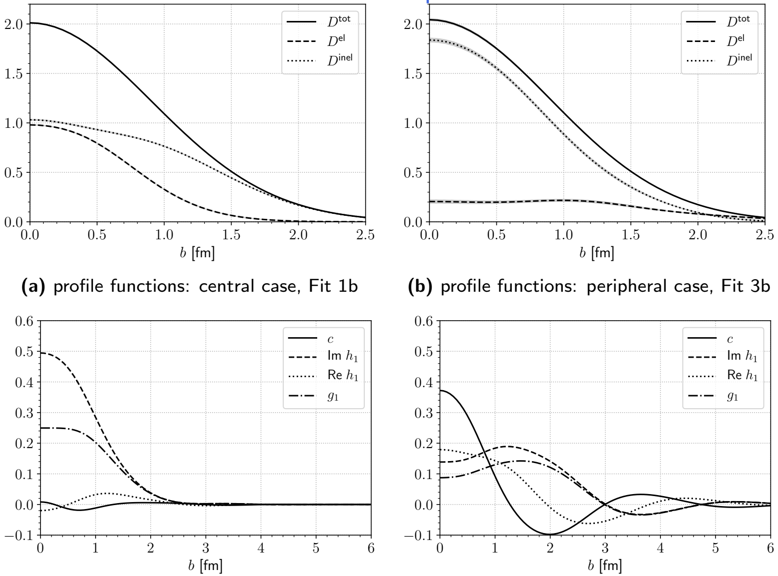


(c) hadronic phases - Fit 3b vs. WY



(d) diffractive slopes - Fit 3b vs. WY

Results at 8 TeV - b-dependent functions



(a) profile functions: central case, Fit 1b

(b) profile functions: peripheral case, Fit 3b

(c) central case - Fit 1b

(d) peripheral case - Fit 3b

References I

[1] A. Andreazza et al., "What Next: White Paper of the INFN-CSN1", Frascati Phys. Ser. **60**, 1–302 (2015).

[2] H. A. Bethe, "Scattering and polarization of protons by nuclei", Ann. Phys. **3**, 190–240 (1958).

[3] G. B. West and D. R. Yennie, "Coulomb interference in high-energy scattering", Phys. Rev. **172**, 1413–1422 (1968).

[4] M. P. Locher, "Relativistic treatment of structure in the coulomb interference problem", Nucl. Phys. B **2**, 525–531 (1967).

[5] V. Kundrát, M. Lokajíček, and I. Vrkoč, "Limited validity of West and Yennie integral formula for elastic scattering of hadrons", Phys. Lett. **B656**, 182–185 (2007).

[6] J. Procházka and V. Kundrát, "Eikonal model analysis of elastic proton-proton collisions at 52.8 GeV and 8 TeV", arXiv:1606.09479 (2019).

[7] J. Kašpar, V. Kundrát, M. Lokajíček, and J. Procházka, "Phenomenological models of elastic nucleon scattering and predictions for LHC", Nucl.Phys. **B843**, 84–106 (2011).

[8] A. Martin, "A theorem on the real part of the high-energy scattering amplitude near the forward direction", Phys. Lett. B **404**, 137–140 (1997).

[9] V. Kundrát, M. Lokajíček, and M. V. Lokajíček, "Are elastic collisions central or peripheral?", Czech. J. Phys. **B31**, 1334 (1981).

[10] TOTEM Collaboration, "Measurement of elastic pp scattering at $\sqrt{s} = 8$ TeV in the Coulomb-nuclear interference region – determination of the ρ-parameter and the total cross-section", Eur. Phys. J. C **76**, 661 (2016), see also CERN-PH-EP-2015-325, arXiv:1610.00603.

[11] TOTEM Collaboration, "Evidence for non-exponential elastic proton-proton differential cross section at low $|t|$ and $\sqrt{s} = 8$ TeV by TOTEM", Nucl. Phys. B **899**, 527–546 (2015), see also CERN-PH-EP-2015-083, arXiv:1503.08111.

[12] TOTEM Collaboration, "Measurement of proton-proton elastic scattering and total cross-section at $\sqrt{s} = 7$ TeV", Europhys. Lett. **101**, 21002 (2013).

[13] J. Procházka, "Elastic proton-proton collisions at high energies", CERN-THESIS-2018-294, PhD thesis (Feb. 2018).

References II

[14] J. Procházka, M. V. Lokajíček, and V. Kundrát, "Dependence of elastic hadron collisions on impact parameter", Eur. Phys. J. Plus **131**, 147 (2016), see also arXiv:1509.05343 (2015).

[15] J. Procházka, V. Kundrát, and M. V. Lokajíček, "Elastic scattering of hadrons without optical theorem", arXiv: 1502.00468 (2015).

[16] M. V. Lokajíček, V. Kundrát, and J. Procházka, "Schrödinger equation and (future) quantum physics", in *Advances in quantum mechanics*, edited by P. Bracken (InTech Publisher, 2013), pp. 105–132.

[17] M. V. Lokajíček and J. Procházka, "The contemporary state of fundamental physical research and the future path to scientific knowledge", arXiv:1610.08331 (2019).

[18] TOTEM Collaboration, "Proton-proton elastic scattering at the LHC energy of $\sqrt{s} = 7$ TeV", Europhys. Lett. **95**, 41001 (2011).

[19] V. Kundrát, M. V. Lokajíček, and D. Krupa, "Impact parameter structure derived from elastic collisions", Phys. Lett. B **544**, 132–138 (2002).

[20] V. Kundrát, M. Lokajíček, J. Kašpar, and J. Procházka, "Problems of phenomenological description of elastic pp scattering at the LHC; predictions of contemporary models", in Proceedings of the Workshop Forward Physics at the LHC, La Biodola, Elba Island, Italy, May 27-29, 2010, edited by F. Ferro and S. Lami, arXiv: hep-ex/1012.5169 (2010), pp. 26–35.

[21] U. Amaldi et al., "Precision measurement of proton-proton total cross section at the cern intersecting storage rings", Nucl. Phys. B **145**, 367–401 (1978).

[22] E. H. De Groot and H. I. Miettinen, "Shadow approach to diffraction scattering", Invated talk presented at the VIIIth Recontre de Moriond, Merible-les-Allues 1973, RL-73-003 (1973).

[23] H. Miettinen, "Impact structure of diffraction scattering", CERN-TH-1864 (1974).

[24] H. Miettinen, "Geometrical description of hadronic collisions", Acta Physica Polonica **B6**, 625 (1975).