The some solution of the Boltzmann equation

Oleksii Hukalov[1] and Vyacheslav Gordevskyy[2]

[1)]B. Verkin ILTPE of NASU, Kharkiv, Ukraine, hukalov@ilt.kharkov.ua

[2)]V. N. Karazin Kharkiv National University, Kharkiv, Ukraine, gordevskyy2006@gmail.com

The Boltzmann equation [1] that describes the evolution of rarefied gases is one of the main equations of the kinetic theory of gases. For a model of hard spheres, the equation has the form:

$$D(f) = Q(f, f), \tag{1}$$

where the left-hand side of the equation is the differential operator:

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(V, \frac{\partial f}{\partial x}\right), \tag{2}$$

and the right-hand side of (1) is the collision integral, which for the hard spheres model is as follows:

$$Q(f, f) \equiv \frac{d^2}{2} \int_{R^3} dV_1 \int_{\Sigma} d\alpha |(V - V_1, \alpha)| \Big[f(t, x, V_1') f(t, x, V') - f(t, x, V) f(t, x, V_1) \Big], \tag{3}$$

where $f(t, x, V)$ is the distribution function of particles.

The solution to this equation will be look for in the next form:

$$f(t, x, V) = \sum_{i=1}^{\infty} \varphi_i(t, x) M_i(t, x, V), \tag{4}$$

where

$$M_i(t, x, V) = \rho_i \left(\frac{\beta_i}{\pi}\right)^{3/2} e^{-\beta_i (V - \overline{V}_i)^2}. \tag{5}$$

The density

$$\rho_i = \rho_{0i} e^{\beta_i \omega_i^2 r_i^2}, \tag{6}$$

where ρ_{0i} is a nonnegative scalar constant, the parameter β_i is the quantity inverse to the absolute temperature:

$$\beta_i = \frac{1}{2T_i}, \tag{7}$$

the vector ω_i is the angular velocity of the gas flow as a whole with which it rotates about some axis, and r_i^2 is the distance between the molecule and the axis of rotation x_{0i}:

$$r_i^2 = \frac{1}{\omega_i^2} [\omega_i, x - x_{0i} - u_{0i} t]^2, \tag{8}$$

$$x_{0i} = \frac{1}{\omega_i^2} [\omega_i, \widehat{V}_i - u_{0i}], \tag{9}$$

(here $[a, b]$ is the vector product of the vectors a and b) the vector $u_{0i} \perp \omega_i$ and it is the linear velocity of the axis of the $i-$th rotating gas flow. By $\widehat{V}_i$, we denote the translational velocity of the flow included in the mass velocity:

$$\overline{V}_i = \widehat{V}_i + [\omega_i, x - u_{0i} t]. \tag{10}$$

The coefficient functions $\varphi_i(t, x)$ are nonnegative smooth functions on $\mathbb{R}^4$ and their norm:

$$||\varphi_i(t, x)|| = \sup_{(t,x) \in \mathbb{R}^4} \left(|\varphi_i(t, x)| + \left| \frac{\partial \varphi_i(t, x)}{\partial t} \right| + \left| \frac{\partial \varphi_i(t, x)}{\partial x} \right| \right) \tag{11}$$

is not equal to zero.

The aim of the work is to find the form of the coefficient functions $\varphi_i(t,x)$ and the conditions for the hydrodynamic parameters of Maxwellians for which the uniform-integral error:

$$\Delta = \Delta(\beta_i) = \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3} \big|D(f) - Q(f,f)\big| dV \tag{12}$$

can be arbitrarily small.

Theorem 1 *Let the coefficient functions have the form*

$$\varphi_i(t,x) = \psi_i(t,x) e^{-\beta_i \omega_i^2 r_i^2}, \tag{13}$$

where $\psi_i(t,x) \geqslant 0$ are smooth nonnegative functions and their norm (11) is not equal to zero. We require that all function series with one of the following common terms

$$\psi_i, \quad |x|\psi_i, \quad t\psi_i, \quad \left|\frac{\partial \psi_i}{\partial x}\right|, \quad \left|\frac{\partial \psi_i}{\partial t}\right|, \quad |x|\left|\frac{\partial \psi_i}{\partial x}\right|, \quad t\left|\frac{\partial \psi_i}{\partial t}\right| \tag{14}$$

converge uniformly on the $\mathbb{R}^4$ after multiplying by ρ_{0i}. Suppose that

$$\omega_i = \omega_{0i}\beta_i^{-m_i}, \; m_i \geqslant \frac{1}{4}. \tag{15}$$

Then there exists a function Δ' such that

$$\Delta \leqslant \Delta', \tag{16}$$

and:

1. *if $m_i > \frac{1}{2}$, then:*

$$\lim_{\beta_i\to+\infty} \Delta' = \sum_{i=1}^{\infty} \rho_{0i} \sup_{(t,x)\in\mathbb{R}^4} \left|\frac{\partial \psi_i}{\partial t} + \left(\widehat{V}_i, \frac{\partial \psi_i}{\partial x}\right)\right| + 2\pi d^2 \sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j} \left|\widehat{V}_i - \widehat{V}_j\right| \sup_{(t,x)\in\mathbb{R}^4} (\psi_i\psi_j); \tag{17}$$

2. *if $m_i = \frac{1}{2}$, then in the right-hand side of (17) there is the additional term:*

$$\frac{4}{\sqrt{\pi}} \sum_{i=1}^{\infty} \rho_{0i} \left|\left[\omega_{0i}, \widehat{V}_i - u_{0i}\right]\right| \sup_{(t,x)\in\mathbb{R}^4} \psi_i; \tag{18}$$

3. *if $\frac{1}{4} < m_i < \frac{1}{2}$ and vectors $\omega_{0i}, \left(\widehat{V}_i - u_{0i}\right)$ are parallel*

$$\omega_{0i} \parallel \left(\widehat{V}_i - u_{0i}\right), \tag{19}$$

   then assertion (17) is also true;

4. *if $m_i = \frac{1}{4}$ and*

$$\omega_i = \omega_{0i} s_i \beta_i^{-\frac{1}{4}}, \tag{20}$$

   where s_i are positive constants and, in addition, require the validity of (19), then in the right-hand side of (17) there is the additional term:

$$\frac{4}{\sqrt{\pi}} \sum_{i=1}^{\infty} \rho_{0i} s_i^2 \omega_{0i}^2 \sup_{(t,x)\in\mathbb{R}^4} \left(\left(|x| + |x - u_{0i}t|\right)\psi_i\right). \tag{21}$$

The sufficient conditions for minimizing the deviation (12) are presented in the following corollary.

Corollary 1 *Let the functions $\psi_i(t,x)$ have the form:*

$$\psi_i(t,x) = C_i\left(x - \widehat{V}_i t\right) \tag{22}$$

or

$$\psi_i(t,x) = E_i\left(\left[x, \widehat{V}_i\right]\right), \tag{23}$$

and let the functions C_i and E_i satisfy the conditions of Theorem 1.

In addition, if one of the following conditions is true:

$$\overline{V}_i = \overline{V}_j, \tag{24}$$

$$\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset \quad (i \neq j), \tag{25}$$

$$d \to 0, \tag{26}$$

then, for $m_i > \frac{1}{2}$, the error (12) can be made arbitrarily small. For $m_i \in \left(\frac{1}{4}, \frac{1}{2}\right]$, (12) is infinitesimally small if the condition (19) of Theorem 1 is fulfilled. For the value $m_i = \frac{1}{4}$, we will also require that

$$s_i \to +0. \tag{27}$$

Below is a theorem that contains another approach for obtaining coefficient functions in the distribution (4).

Theorem 2 *Let all function series with a common term of (14) after multiplying by a factor $e^{\beta_i \omega_i^2 r_i^2}$ retain the convergence uniformly on $\mathbb{R}^4$. We also assume that the condition (15) remains true and (19) is valid.*

Then there exists a value Δ', for which (16) is true, and

$$\begin{aligned}\lim_{\beta_i \to +\infty} \Delta' = \sum_{i=1}^{\infty} \rho_{0i} \sup_{(t,x)\in\mathbb{R}^4} \left(\mu_i(t,x)\left|\frac{\partial\varphi_i}{\partial t} + \left(\widehat{V}_i, \frac{\partial\varphi_i}{\partial x}\right)\right|\right) \\ + 2\pi d^2 \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \rho_{0i}\rho_{0j}\left|\widehat{V}_i - \widehat{V}_j\right| \sup_{(t,x)\in\mathbb{R}^4} (\mu_i(t,x)\mu_j(t,x)\varphi_i\varphi_j),\end{aligned} \tag{28}$$

where:

$$\mu_i(t,x) = \begin{cases} e^{[\omega_{0i}, x - u_{0i}t]^2}, & m_i = \frac{1}{2} \\ 1, & m_i > \frac{1}{2} \end{cases}.$$

REFERENCES.

1. S. Chapman and T.G. Cowling. The Mathematical Theory of Non-Uniform Gases. // Cambridge Univ. Press, Cambridge, 1952.

2. Hukalov, O. O.; Gordevskyy, V. D. The Interaction of an Infinite Number of Eddy Flows for the Hard Spheres Model. // J. Math. Phys. Anal. Geom. 2021, 17, 163-174.