

Representations of solutions for a first order quasilinear equation in a form of series in powers of the copolynomial δ -function

R. V. Skurikhin¹ S. L. Gefter² A. L. Piven'¹

¹V. N. Karazin Kharkiv National University, Kharkiv, Ukraine

²B. Verkin Institute for Low Temperature Physics and Engineering of the NAS of Ukraine, Kharkiv, Ukraine

THE CAUCHY PROBLEM (P)

$$u_t + f(u)u_x = 0, \quad u(0, x) = \delta(x)$$

An algebraic approach to singular initial data.

Polynomial tests make multiplication explicit and reveal combinatorial coefficients.

01 / THE IDEA

Polynomials as test functions

The nonlinear term $f(u)u_x$ requires a product. For classical distributions, such a product is not defined in general.

What is a copolynomial?

A generalized function is described by how it acts on test functions. Here we choose **polynomials as our test functions**. A **copolynomial** $T \in \mathbb{R}[x]'$ assigns a real number to each polynomial, linearly:

$$T : \mathbb{R}[x] \longrightarrow \mathbb{R}, \quad \langle T, p \rangle = T(p),$$

The simplest example: evaluation

The copolynomial δ evaluates a polynomial at zero. For $p(x) = a_0 + a_1x + \dots + a_nx^n$,

$$\langle \delta, p \rangle = p(0) = a_0.$$

More generally, the moments $m_k = \langle T, x^k \rangle$ determine how T acts on every polynomial:

$$\langle T, p \rangle = \sum_{k=0}^n a_k m_k.$$

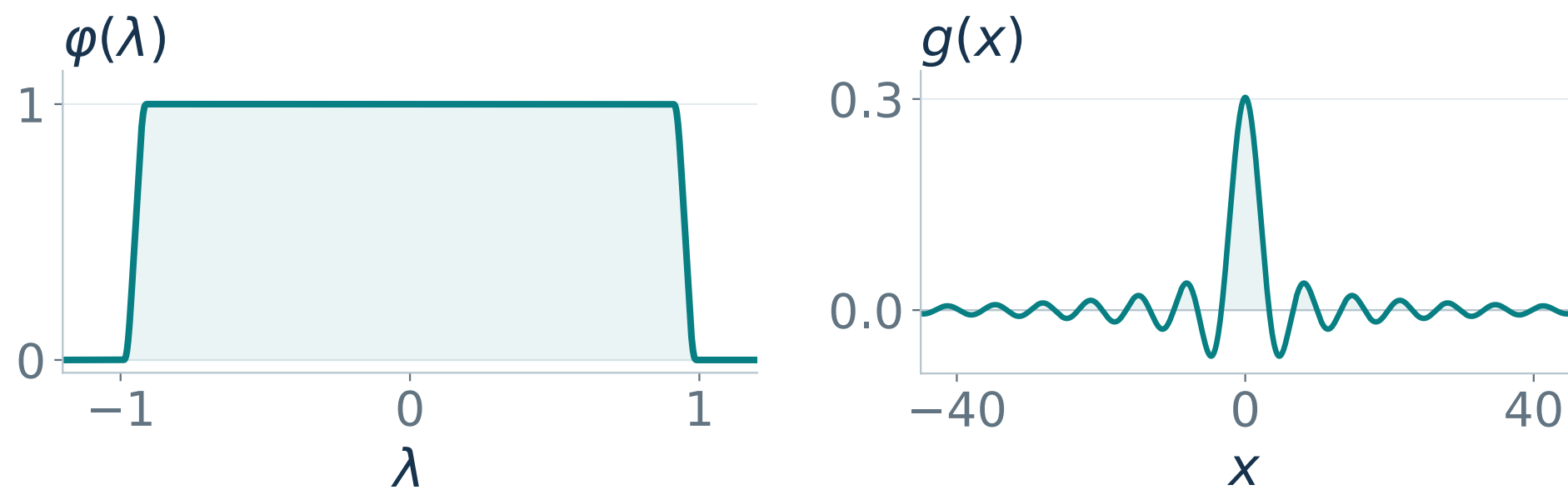
Two copolynomials are equal when they agree on every polynomial.

Functions can represent copolynomials

An integrable function g with all absolute moments gives another way to assign numbers to polynomials:

$$\langle T_g, p \rangle = \int_{\mathbb{R}} p(x)g(x) dx.$$

A smooth cutoff φ , equal to one near zero (left), gives $g = \mathcal{F}^{-1}\varphi$ (right) by inverse Fourier transformation.



Since $\varphi(0) = 1$ and $\varphi^{(k)}(0) = 0$ for $k \geq 1$, the Fourier moment identities give

$$\int_{\mathbb{R}} g(x) dx = 1, \quad \int_{\mathbb{R}} x^k g(x) dx = 0 \quad (k \geq 1).$$

When we integrate a polynomial against g , **only its constant term survives**:

$$\int_{\mathbb{R}} p(x)g(x) dx = a_0 = p(0).$$

Hence $T_g = \delta$ as copolynomials.

02 / THE CONSTRUCTION

Make multiplication explicit

Encode a functional by its moments

Collect the moments $m_k = \langle T, x^k \rangle$ in the **Cauchy–Stieltjes transform**:

$$C(T)(s) = \sum_{k=0}^{\infty} \frac{\langle T, x^k \rangle}{s^{k+1}}.$$

This is a one-to-one correspondence with **formal Laurent series at infinity** containing only strictly negative powers of s . In particular,

$$C(\delta)(s) = \frac{1}{s}.$$

Multiply the series; transform back

$$\begin{array}{ccc} T_1, T_2 & \xrightarrow{C} & C(T_1), C(T_2) \\ & & \downarrow \times \\ T_1 T_2 & \xleftarrow{C^{-1}} & C(T_1)C(T_2) \end{array}$$

$$T_1 T_2 = C^{-1}(C(T_1)C(T_2)).$$

The product is commutative and associative.

Differentiation still works

Define derivatives as for classical distributions:

$$\langle T', p \rangle = -\langle T, p' \rangle.$$

The transform respects differentiation,

$$C(T') = \frac{d}{ds} C(T),$$

so the product satisfies the Leibniz rule:

$$(T_1 T_2)' = T_1' T_2 + T_1 T_2'.$$

Now powers of δ have a meaning

$$\boxed{\delta^2 = -\delta'} \quad \delta^{n+1} = \frac{(-1)^n}{n!} \delta^{(n)}.$$

Here δ^m is an **algebraic power**; $\delta^{(n)}$ is a derivative.

Every copolynomial has the expansion

$$T = \sum_{n=0}^{\infty} \langle T, x^n \rangle \delta^{n+1}.$$

On each polynomial test, only finitely many terms contribute.

03 / THE RESULT

From δ to Catalan numbers

We seek a formal power series in t :

$$u(t) = \sum_{k \geq 0} u_k t^k, \quad u_k \in \mathbb{R}[x]'. \quad \text{For } f(z) = \sum_{j=1}^{\ell} f_j z^j \text{ with } f(0) = 0, \text{ problem (P) has a unique formal solution in the ring } \mathbb{R}[x]'[[t]].$$

For $f(z) = \sum_{j=1}^{\ell} f_j z^j$ with $f(0) = 0$, problem **(P)** has a **unique formal solution** in the ring $\mathbb{R}[x]'[[t]]$.

The simplest nonlinear case: $f(u) = u$

$$u_t + u u_x = 0, \quad u(0) = \delta.$$

The solution starts with

$$u(t) = \delta + \delta^3 t + 2\delta^5 t^2 + 5\delta^7 t^3 + 14\delta^9 t^4 + \dots$$

Its coefficients are the **Catalan numbers**:

$$C_k = \frac{1}{k+1} \binom{2k}{k}, \quad u(t) = \sum_{k=0}^{\infty} C_k \delta^{2k+1} t^k.$$

Monomial nonlinearities

More generally, for $f(u) = u^{\ell}$ with $\ell \geq 1$,

$$u(t) = \sum_{k=0}^{\infty} A_k(1, \ell+1) \delta^{(\ell+1)k+1} t^k.$$

Here the coefficients are the **Fuss–Catalan–Raney numbers**:

$$A_k(1, \ell+1) = \frac{((\ell+1)k)!}{k!(\ell k+1)!}.$$

The case $\ell = 1$ recovers the Catalan example above:

$$A_k(1, 2) = C_k = 1, 1, 2, 5, 14, \dots$$

The general formula

For $\alpha \in \mathbb{N}_0^{\ell}$, write

$$|\alpha| = \sum_j \alpha_j, \quad w(\alpha) = \sum_j j \alpha_j, \quad \alpha! = \prod_j \alpha_j!, \quad f^{\alpha} = \prod_j f_j^{\alpha_j}.$$

Then the solution of **(P)** is

$$u(t) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{f^{\alpha} (w(\alpha) + k)!}{\alpha! (w(\alpha) + 1)!} \delta^{w(\alpha)+k+1} t^k.$$

For every $t \in \mathbb{R}$, this series **converges in** $\mathbb{R}[x]'$.

On each polynomial test, only finitely many terms contribute.

Polynomial tests $\longrightarrow$ a differential algebra $\longrightarrow$ explicit nonlinear evolution

BACKGROUND AND REFERENCES

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